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................**ORDINARY DIFFERENTIAL EQUATIONS**................

Solving Variational Inequalities with Coupling Constraints with the Use of Differential Equations

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1. STATEMENT OF THE PROBLEM

We consider the following variational inequality with coupling constraints:

$$\langle F(v^*), w - v^* \rangle \geq 0 \quad \forall w \in \Omega_0, \ \ g(v^*, w) \leq 0, \tag{1.1}$$

where $F(v) : R^n \to R^n$, $g(v, w) : R^n \times R^n \to R^m$, and $\Omega_0 \in R^n$ is a closed convex set.

The presence of functional constraints of the form $g(v, w) \leq 0$, which couple the parameters and the variables of the problem, is the basic distinction of this statement from the standard one. The coupling constraints make these problems difficult to solve. However, numerous mathematical models contain coupling constraints, which is the reason for the interest in such problems. Methods for solving variational inequalities with coupling constraints have been considered only in few papers, of which we note the paper [1]. On the contrary, there is an extensive literature (e.g., see the review [2]) dealing with the standard statement of problems involving variational inequalities, including solution methods.

Problems with coupling constraints arise in numerous fields of mathematics. First, we note economic equilibrium models, which, by definition, always contain budget constraints implying that the inner product of the price vector by the commodity vector does not exceed some a priori given expenditures. By their very nature, these constraints are always coupling [3]. Generalized statements of n-person games also result in variational inequalities with coupling constraints [4]. Coupling constraints naturally occur in problems of equilibrium programming [5] and hierarchical programming [6]. The development of this direction in applied mathematical physics, where variational inequalities appeared for the first time, leads to inequalities with coupling constraints [7]. This brief list of problems shows that coupling constraints are not specific to some particular problem. Quite opposite, they are typical of a wide class of problems. Therefore, it is topical to develop methods for solving problems with coupling constraints. In the present paper we suggest and justify the convergence (asymptotic stability) of a trajectory of a feedback-controlled differential gradient system to a solution of a variational inequality with coupling constraints.

2. TOPICAL PROBLEMS

Here we briefly outline the best-known problems in which coupling constraints arise intrinsically.

2.1. A Two-Person Game with Coupling Constraints

To simplify the exposition, we consider the following two-person game with scalar coupling constraints [4, 8]:

$$\begin{aligned} x_1^* &\in \operatorname{Argmin}\{f_1(x_1, x_2^*) \mid g_1(x_1, x_2^*) \le g_1(x_1^*, x_2^*), \;\; x_1 \in Q_1\}, \\ x_2^* &\in \operatorname{Argmin}\{f_2(x_1^*, x_2) \mid g_2(x_1^*, x_2) \le g_2(x_1^*, x_2^*), \;\; x_2 \in Q_2\}, \end{aligned} \tag{2.1}$$

where $f_1, f_2, g_1, g_2 : R^n \times R^n \to R^1$. Here each function is convex with respect to the variables with the same subscript for arbitrary values of the remaining variables; i.e., f_1 and g_1 are convex in x_1 for any x_2, and f_2 and g_2 are convex in x_2 for any x_1.

Each n-person game can always be scalarized and reduced to the problem of finding a fixed point of an extremal mapping. This procedure was described for the first time in [9] for a game without coupling functional constraints. However, this approach can also be applied to games with coupling constraints. Let us show that the reduction is possible. We introduce two normalized functions of the form $\Phi(v, w) = f_1(x_1, y_2) + f_2(y_1, x_2)$ and $G(v, w) = g_1(x_1, y_2) + g_2(y_1, x_2)$, where $v = (y_1, y_2)$, $w = (x_1, x_2)$, and $v, w \in \Omega_0 = Q_1 \times Q_2$. Using these functions, we state the problem as follows. Find a vector $v^* \in \Omega_0$ such that

$$v^* \in \operatorname{Argmin}\{\Phi(v^*, w) \mid G(v^*, w) - G(v^*, v^*) \le 0 \;\; w \in \Omega_0\}. \tag{2.2}$$

Let us show that each solution of problem (2.2) is a solution of problem (2.1) as well.

Indeed, problem (2.2) is equivalent to the validity of the inequality

$$f_1(x_1^*, x_2^*) + f_2(x_1^*, x_2^*) \le f_1(x_1, x_2^*) + f_2(x_1^*, x_2)$$

for all x_1 and x_2 such that

$$g_1(x_1, x_2^*) + g_2(x_1^*, x_2) - g_1(x_1^*, x_2^*) - g_2(x_1^*, x_2^*) \le 0$$

for all $x_1 \in Q_1$ and $x_2 \in Q_2$. In particular, this system of inequalities is valid for all pairs $x_1, x_2^* \in Q_1 \times x_2^*$. The latter means that in this case the system in question acquires the form $f_1(x_1^*, x_2^*) \le f_1(x_1, x_2^*)$ for all x_1 and x_2 such that $g_1(x_1, x_2^*) \le g_1(x_1^*, x_2^*)$ for all $x_1 \in Q_1$. Since this set contains the point x_1^*, it obviously follows that the last system of inequalities is equivalent to the first problem in (2.1). Similar considerations performed for the point (x_1^*, x_2^*) imply the second problem in (2.1).

We can readily see that if the objective function of problem (2.2) is differentiate, then this problem can always be represented as the variational inequality $\langle \nabla_w \Phi(v^*, v^*), w - -v^* \rangle \ge 0$ for all $w \in \Omega_0$, $G(v^*, w) \le G(v^*, v^*)$, where $\nabla_w \Phi(v, v) = \nabla_w \Phi(v, w)|_{v=w}$.

2.2. The Simplest Model of Price Equilibrium

Let us consider the simplest market with a single aggregated consumer [10]. Let $f(x)$ be his utility function, β a given amount of money, and x the demand vector. The cost of the commodities is described by the price vector p. The situation is as follows: on the one hand, the consumer cannot buy commodities of total cost $> \beta$; on the other hand, none of the components of the purchase vector can exceed the stock vector y_0. Therefore, assuming that the consumer buys commodities so as to maximize his utility function, we arrive at the following problem: find a vector $p = p^*$ of equilibrium prices and an optimal demand vector $x = x^*$ such that

$$x^* \in \operatorname{Argmax}\{f(x) \mid \langle p^*, x\rangle \le \beta,\ \ x \in Q\}, \qquad x^* \le y_0. \tag{2.3}$$

If, in this problem, we strengthen the material balance condition $x^* \le y_0$ by the financial balance condition $\langle p^*, x^* - y_0\rangle = 0$, then the set of these conditions satisfies the inequality

$$\langle p - p^*, x^* - y_0\rangle \le 0$$

for all $p \ge 0$. This means that the nonpositive linear functional $\langle p - p^*, x^* - y_0\rangle$ attains its maximum on the positive octant at the point p^*. In other words, we obtain the problem

$$x^* \in \operatorname{Argmax}\{f(x) \mid \langle p^*, x\rangle \le \beta,\ \ x \in Q\}, \qquad p^* \in \operatorname{Argmax}\{\langle p - p^*, x^* - y_0\rangle \mid p \ge 0\},$$

whose solution is a solution of problem (2.3). The problem thus obtained is a problem of the form (2.1).

On the market in question, the aggregated manufacturer is represented by the vector y_0. But its presence on the market can be substantially strengthened. To this end, we allow it to minimize the production of commodities that will never be bought at given prices. We have thereby obtained the model

$$\begin{aligned} &x^* \in \operatorname{Argmax}\{f(x) \mid \langle p^*, x\rangle \le \beta,\ \ x \in Q\},\\ &y^* \in \operatorname{Argmin}\{\langle p^*, y\rangle \mid x^* \le y,\ \ y \in Y\}, \end{aligned} \tag{2.4}$$

where Y is the set of feasible plans of the manufacturer. In the general case, for arbitrary prices p, the feasible set of the manufacturer can be empty; therefore, in the problem, one must choose prices $p = p^*$ such that $\{y \mid x^* \le y,\ \ y \in Y\} \ne \emptyset$, which provides the existence of a solution.

2.3. A Multicriterial Model for Decision Making on a Subset of Efficient Points

In a multicriterial decision-making problem [11], there is a set Q of feasible solutions x on which a vector efficiency criterion $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ is defined. The decision maker tries to increase each of the scalar criteria on a given set of alternatives. In the convex case, the scalarization of the vector criterion according to the formula $\langle \lambda, f(x)\rangle = \sum_{i=1}^{m} \lambda_i f_i(x)$ permits one to describe the set of optimal alternatives (the Pareto set) as the set of optimal solutions of the family of scalar optimization problems $x_\lambda \in$ $\in \operatorname{Argmax}\{\langle \lambda, f(x)\rangle \mid x \in Q\}$ [12]. In the general case of a multicriterion decision-making problem, one must choose a parameter value $\lambda = \lambda^*$ and the corresponding optimal solution x^* so that both vectors belong to a given subset of efficient points, i.e.,

$$x^* \in \operatorname{Argmax}\{\langle \lambda^*, f(x)\rangle \mid x \in Q\}, \qquad g(x^*, \lambda^*) \le 0. \tag{2.5}$$

Assuming that the vectors λ and $g(x, \lambda)$ have the same dimension and strengthening the requirement $g(x^*, \lambda^*) \le 0$ by the additional condition $\langle \lambda, g(x^*, \lambda^*) \rangle = 0$, just as for (2.3), we obtain the following problem, whose solution is also a solution of problem (2.5):

$$x^* \in \operatorname{Argmax}\{\langle \lambda^*, f(x) \rangle \mid x \in Q\}, \qquad \lambda^* \in \operatorname{Argmax}\{\langle \lambda - \lambda^*, g(x^*, \lambda^*) \rangle \mid \lambda \ge 0\}.$$

This problem has the form (2.1).

If the model (2.5) describes a large technical project, then the maximization of the vector criterion provides the project efficiency, while the conditions $g(x, \lambda) \le 0$ describe financial, ecological, and other constraints.

2.4. Quasivariational Inequalities

Let us consider a bilinear two-person game with coupling constraints specified by a closed convex set $K \subset Q_1 \times Q_2$ belonging to $R^n \times R^n$ (see [7]). Through an arbitrary given point $\bar{x} = (\bar{x}_1, \bar{x}_2) \in K$, we draw two cross-sections of the form $K_1(\bar{x}) = \{x_1 \in R^n \mid$ $\mid (x_1, \bar{x}_2) \in K\}$ and $K_2(\bar{x}) = \{x_2 \in R^n \mid (\bar{x}_1, x_2) \in K\}$ and consider the game

$$\begin{aligned} x_1^* &\in \operatorname{Argmin}\{\langle A_1 x_1, x_2^* \rangle + \langle l_1, x_1 \rangle \mid x_1 \in K_1(x^*)\}, \\ x_2^* &\in \operatorname{Argmin}\{\langle x_1^*, A_2 x_2) \rangle + \langle l_2, x_2 \rangle \mid x_2 \in K_2(x^*)\}, \end{aligned} \tag{2.6}$$

where $x^* = (x_1^*, x_2^*)$. If we introduce the matrix $A^\top$, where $\top$ stands for transposition, with entries $a_{11} = 0$, $a_{12} = A_1^\top$, $a_{21} = A_2^\top$, and $a_{22} = 0$ and the vector $l = (l_1, l_2)$, then we can represent problem (2.6) in the equivalent form of the variational inequality

$$\langle A^\top x^*, x - x^* \rangle + \langle l, x - x^* \rangle \ge 0 \quad \forall x \in K(x^*), \tag{2.7}$$

where $K(x^*) = K_1(x^*) \times K_2(x^*)$.

If $A_1^\top$ and $A_2^\top$ are differential operators, and, moreover, $K \in Q_1 \times Q_2 \subseteq H^1 \times H^2$, where H^1 and H^2 are Hilbert spaces, then problem (2.7) is referred to as a quasivariational inequality [7, p.244].

Note that if $g_1(x_1, x_2) = g_2(x_1, x_2)$ in problem (2.1), then this problem acquires the form (2.6).

2.5. Two-Level Programming

The ordinary maximin problem can be treated as the simplest problem of hierarchical programming [5]. Indeed, in the problem of finding an optimal strategy maximizing the minimum function, namely,

$$\max_x \{\min_y f(x, y) \mid g(x, y) \le 0,\ \ y \in Y\} = \max_x \min_y \{f(x, y) \mid g(x, y) \le 0,\ \ y \in Y\}.$$

where $x \in X(y) \subset R^n$ and $y \in Y \subset R^n$, any point of the manifold

$$y(x) = \operatorname{Argmin}\{f(x, y) \mid g(x, y) \le 0,\ \ y \in Y\}$$

can be a solution of this problem. However, if $f(x, y)$ and $g(x, y)$ are convex in y for each x and x^* is a fixed point of the extremal inclusion $x^* \in \operatorname{Argmin}\{f(x^*, y) \mid g(x^*, y) \le$ $\le 0,\ \ y \in Y\}$, then this minimax problem can be reduced to finding a fixed point of this extremal mapping.

3. SYMMETRIC FUNCTIONS

Problem with coupling constraints have always been attracting the attention of researchers. We note the papers [1, 13], where gradient approaches to such problems were discussed. Game problems with coupling constraints were considered in [14]. In these works, it was assumed that the function $g(v, w)$ specifying the constraints is jointly convex in (v, w). This is quite a restrictive requirement, which is never valid for constraints used in economic equilibrium models since they include budget constraints of the form $\langle p, x\rangle \le m$, where p is the price vector, x is the commodity vector, and m is given expenditures. Here the function $g(p, x) = \langle p, x\rangle$ is not jointly convex.

In the present paper, we omit the requirement of simultaneous convexity of the function $g(v, w)$ with respect to the variables v and w and use the symmetry of this function with respect to the diagonal $v = w$ of the square $\Omega_0 \times \Omega_0$.

We introduce the following notion.

Definition 1. *A function $g(v, w) : I\!R^n \times I\!R^n \to I\!R^m$ is said to be symmetric on $R^n \times R^n$ if*

$$g(v, w) = g(w, v) \qquad \forall w \in \Omega_0, \quad \forall v \in \Omega_0. \tag{3.1}$$

One can readily give examples of symmetric functions. First, we note functions inducing budget constraints of the form $g(v, w) = \langle v, w\rangle$ and $g(v, w) = \langle Av, w\rangle$ in economic equilibrium models, where A is a symmetric matrix. The Cobb–Douglas production function $g(v, w) = Av^\alpha w^\beta$ and the CES function $g(v, w) = A(\alpha v^{-\omega} + \beta w^{-\omega})^{-\gamma/\omega}$, where $A > 0$, $\alpha > 0$, $\beta > 0$, $\omega > 0$ are parameters, are well known in applications. If $\alpha = \beta$, then these functions are symmetric in the sense of (3.1). We can readily see that the function $\Phi(v, w) = f(x_1, y_2) + f(y_1, x_2)$, where $v = (y_1, y_2)$ and $w = (x_1, x_2)$, is symmetric.

Let us find characteristic properties of symmetric functions [15]. To this end, we differentiate (3.1) with respect to the variable w:

$$\nabla_w^\top g(v, w) = \nabla_v^\top g(w, v) \quad \forall w \in \Omega_0, \quad \forall v \in \Omega_0, \tag{3.2}$$

where $\nabla_w^\top g(v, w)$ and $\nabla_v^\top g(w, v)$ are $(m\times n)$-matrices whose rows are the vectors $\nabla_v g_i(w, v)$ and $\nabla_w g_i(v, w)$, $i = 1, 2, \ldots, m$.

By setting $w = v$ in (3.2), we obtain

$$\nabla_w^\top g(v, v) = \nabla_v^\top g(v, v) \quad \forall v \in \Omega_0. \tag{3.3}$$

Therefore, we can state the following assertion.

Property 1. *The restrictions of the matrices of the gradient of a symmetric vector function $g(v, w)$ with respect to the variables v and w to the diagonal of the square $\Omega_0 \times \Omega_0$ are equal to each other.*

By the definition of differentiability of the function $g(v, w)$, we have [16, p.92]

$$g(v + h, w + k) = g(v, w) + \nabla_v^\top g(v, w)h + \nabla_w^\top g(v, w)k + \omega(v, w, h, k), \tag{3.4}$$

where $\omega(v,w,h,k)/(|h|^2+|k|^2)^{1/2} \to 0$ as $|h|^2+|k|^2 \to 0$. Let $w=v$ and $h=k$. Then from (3.3) and (3.4), we obtain

$$g(v+h, v+h) = g(v,v) + 2\nabla_w^\top g(v,v)h + \omega(v,h), \tag{3.5}$$

where $\omega(v,h)/|h| \to 0$ as $|h| \to 0$. Since formula (3.5) is a special case of (3.4), it follows that the restriction of the gradient $\nabla_w^\top g(v,w)$ to the diagonal of the square $\Omega_0 \times \Omega_0$ is the gradient $\nabla^\top g(v,v)$ of the function $g(v,v)$, i.e.,

$$2\nabla_w^\top g(v,w)|_{v=w} = \nabla^\top g(v,v) \quad \forall v \in \Omega_0. \tag{3.6}$$

We have thereby proved the following property.

Property 2 ([17]). *The operator $2\nabla_w g(v,w)|_{v=w}$ is a potential operator and coincides with the gradient of the restriction of the symmetric function $g(v,w)$ to the diagonal of the square, i.e., $2\nabla_w^\top g(v,v) = \nabla^\top g(v,v)$.*

This key property of symmetric functions plays an important role in the forthcoming considerations.

As we have already mentioned, if $g_1(x_1,x_2) = g_2(x_1,x_2)$ in problem (2.1), then this problem can be reduced to the form (2.6). Let us show that in this case the normalized function $G(v,w)$ occurring in (2.2) satisfies the symmetry property (3.1). Indeed,

$$G(v,w) = g_1(x_1,y_2) + g_2(y_1,x_2), \qquad G(w,v) = g_1(y_1,x_2) + g_2(x_1,y_2);$$

but since $g_1(x_1,x_2) = g_2(x_1,x_2)$, we obviously have $G(v,w) = G(w,v)$. Therefore, in our case, problem (2.1) has symmetric coupling constraints.

4. SYMMETRIZATION

However, the coupling constraints in problem (1.1) need not have the symmetry property; for example, they can be antisymmetric, i.e., satisfy the condition $g(v,w) = -g(w,v)$ for all $v,w \in \Omega_0$. Let us show that in this case the coupling constraints do not affect the solution of problem (1.1) and can be omitted altogether. Indeed, let us consider the pair of problems $\langle F(v^*), w-v^*\rangle \geq 0$ for all $w \in \Omega_0$ and $\langle F(v^*), w-v^*\rangle \geq 0$, $g(v^*,w) \leq 0$ for all $w \in \Omega_0$, where $g(v,w)$ is an antisymmetric function. Such a function always vanishes on the diagonal of the square $\Omega_0 \times \Omega_0$. Since the relation $g(v,v) = -g(v,v)$ with $v=w$ implies $g(v,v)=0$, we consider the intersection $\Omega_0 \cap \{w \mid g(v^*,w) \leq 0\}$. This intersection is always nonempty (it contains the point v^*) and is a subset of Ω_0. Since v^* is a point of minimum of the function $\langle F(v^*), w-v^*\rangle$ on Ω_0, i.e., a solution of the first problem, we see that so much the more this point is a point of minimum of this function on an arbitrary subset of this set, i.e., a solution of the second problem. Therefore, antisymmetric coupling constraints in equilibrium problems can be omitted.

In the general case, if $g(v,w)$ is neither symmetric nor antisymmetric, then the constraints of problem (1.1) can be symmetrized. Indeed, let us introduce two subclasses of symmetric and antisymmetric vector functions by setting

$$g(v,w) - g(w,v) = 0 \qquad \forall w \in \Omega_0, \quad \forall v \in \Omega_0, \tag{4.1}$$

$$g(v,w) + g(w,v) = 0 \qquad \forall w \in \Omega_0, \quad \forall v \in \Omega_0. \tag{4.2}$$

These conditions generalize the notions of symmetric and antisymmetric matrices (see [17]). We define the transposed function as $^{\top}(v, w) = g(w, v)$ [15]. Using the notion of a transposed function, we can restate the symmetry condition (4.1) and the antisymmetry condition (4.2) as $\Phi(v, w) = \Phi^{\top}(v, w)$ and $\Phi(v, w) = -\Phi^{\top}(v, w)$. Using the obvious relations

$$\Phi(v, w) = (\Phi^{\top}(v, w))^{\top}, \qquad (\Phi_1(v, w) + \Phi_2(v, w))^{\top} = \Phi_1^{\top}(v, w) + \Phi_2^{\top}(v, w),$$

we can readily see that any real function $\Phi(v, w)$ admits the expansion $g(v, w) = s(v, w) + k(v, w)$, where $s(v, w)$ is a symmetric function and $k(v, w)$ is an antisymmetric function. This expansion is unique; moreover, $s(v, w) = 0.5(g(v, w) + g^{\top}(v, w))$ and $k(v, w) = 0.5(g(v, w) - g^{\top}(v, w))$.

Using this expansion, we represent the functional constraints of problem (1.1) in the form

$$\{w \mid g(v^*, w) = s(v^*, w) + k(v^*, w) \leq 0, \;\; w \in \Omega_0\}.$$

Apparently, it follows from the considerations at the beginning of this section that the antisymmetric part of the constraints can be omitted in this case as well. Indeed, let v^* be a solution of the problem

$$\langle F(v^*), w - v^* \rangle \geq 0, \quad s(v^*, w) \leq 0, \quad w \in \Omega_0. \tag{4.3}$$

We introduce the sets

$$\begin{aligned} D &= \{w \mid g(v^*, w) \leq 0, \quad w \in \Omega_0\}, \\ K_1 &= \{w \mid k(v^*, w) \leq 0, \quad w \in \Omega_0\}, \\ K_2 &= \{w \mid k(v^*, w) > 0, \quad w \in \Omega_0\}. \end{aligned}$$

We split the feasible set D of the original problem into two parts, $D_1 = D \cap K_1$ and $D_2 = D \cap K_2$, moreover $D = D_1 \cup D_2$. For all $w \in D_2$, the quantity $k(v^*, w)$ occurring in the inequality

$$s(v^*, w) + k(v^*, w) \leq 0, \qquad w \in \Omega_0,$$

can be omitted; then we can claim that $D_2 \subset \{w \mid s(v^*, w) \leq 0, \;\; w \in \Omega_0\}$. On the other hand, consider the intersection $D_1 \cap \{w \mid s(v^*, w) \leq 0, \;\; w \in \Omega_0\}$, which contains the solution v^* and on which the function $\langle F(v^*), w - v^* \rangle$ attains its minimum. We can readily see that any point belonging to this intersection satisfies the condition $s(v^*, w) + k(v^*, w) \leq 0$, $w \in \Omega_0$. Consequently, if the solution of problem (1.1) has an interior neighborhood, for example, if $g(v^*, v^*) < 0$, $w \in \Omega_0$, then the solution of problem (4.3) is at the same time the solution of problem (1.1).

Therefore, to find a solution of problem (1.1), we must solve the symmetrized problem

$$\langle F(v^*), w - v^* \rangle, \qquad g(v^*, w) + g^{\top}(v^*, w) \leq 0, \qquad w \in \Omega_0.$$

In principle, the idea of symmetrization of constraints offers a possibility of solving equilibrium problems with coupling constraints. For some ideas on symmetrization of sets, see also [18, p.192].

5. REDUCTION TO A SADDLE PROBLEM

Problem (1.1) can always be considered as the minimization problem for the linear function $f(w) = \langle F(v^*), w - v^* \rangle$ on the set $\Omega = \{w \in \Omega_0 \mid g(v^*, w) \le 0\}$. Consider the Lagrange function $\mathcal{L}(v^*, w, p) = \langle F(v^*), w - v^* \rangle + \langle p, g(v^*, w) \rangle$ for all $w \in \Omega_0$ and $p \ge 0$, where v^* is a solution of the problem and w and p are the primal and dual variables, respectively. Since v^* is a point of minimum of $f(w)$ on Ω, it follows that (under certain regularity conditions) the pair (v^*, p^*) is a saddle point of $\mathcal{L}(v^*, w, p)$, i.e., satisfies the system of inequalities

$$\begin{aligned} \langle F(v^*), v^* - v^* \rangle + \langle p, g(v^*, v^*) \rangle &\le \langle F(v^*), v^* - v^* \rangle + \langle p^*, g(v^*, v^*) \rangle \le \\ &\le \langle F(v^*), w - v^* \rangle + \langle p^*, g(v^*, w) \rangle \end{aligned} \tag{5.1}$$

for all $w \in \Omega_0$ and $p \ge 0$. This system of inequalities can be represented in the form

$$\begin{aligned} &v^* \in \operatorname{Argmin}\{\langle F(v^*), w - v^* \rangle + \langle p^*, g(v^*, w) \rangle \mid w \in \Omega_0\}, \\ &p^* \in \operatorname{Argmax}\{\langle p, g(v^*, v^*) \rangle \mid p \ge 0\}. \end{aligned} \tag{5.2}$$

There also exist other equivalent representations of system (5.1). Assuming the differentiability of $g(v, w)$ with respect to w for each v, we rewrite system (5.2) in the form

$$\begin{aligned} &\langle F(v^*) + \nabla_w^\top g(v^*, v^*) p^*, w - v^* \rangle \ge 0 \qquad &\forall w \in \Omega_0, \\ &\langle -g(v^*, v^*), p - p^* \rangle \ge 0 \qquad &\forall p \ge 0. \end{aligned} \tag{5.3}$$

Using the projection operator, we represent the resulting system of variational inequalities as the operator equations

$$v^* = \pi_{\Omega_0}(v^* - \alpha(F(v^*) + \nabla_w^\top g(v^*, v^*) p^*), \qquad p^* = \pi_+(p^* + \alpha g(v^*, v^*)), \tag{5.4}$$

where $\pi_+(\ldots)$ and $\pi_{\Omega_0}(\ldots)$ are the operators of the projection on the positive octant R^n_+ and the set Ω_0, respectively,] and $\alpha > 0$ is an increment type parameter.

Let us transform the system of inequalities (5.3). The first inequality in this system can be represented as $\langle F(v^*), w - v^* \rangle + \langle p^*, \nabla_w g(v^*, v^*)(w - v^*) \rangle \ge 0$ for all $w \in \Omega_0$. Then, taking into account the key property (3.6) of symmetric functions and the convexity of the function $g(v, w)_{v=w}$ on the diagonal of the square $\Omega_0 \times \Omega_0$, we separately transform the term

$$\langle p^*, \nabla_w g(v^*, v^*)(w - v^*) \rangle = \frac{1}{2} \langle p^*, \nabla g(v^*, v^*)(w - v^*) \rangle \le \frac{1}{2} \langle p^*, g(w, w) - g(v^*, v^*) \rangle.$$

Then relation (5.3) can finally be represented as

$$\begin{aligned} &\langle F(v^*), w - v^* \rangle + 0.5 \langle p^*, g(w, w) - g(v^*, v^*) \rangle \ge 0 \qquad &\forall w \in \Omega_0, \\ &\langle -g(v^*, v^*), p - p^* \rangle \ge 0 \qquad &\forall p \ge 0. \end{aligned} \tag{5.5}$$

The solution of a variational inequality with coupling constraints is thereby reduced to the saddle problem (5.5). Methods for solving this problem were developed in [19]. However, the development of methods in terms of the original problem is quite interesting for two reasons: first, these methods are interpreted as dynamic trade-off models with conflicting objectives; second, they form a basis for various symmetrization procedures for problems with nonsymmetric coupling constraints.

6. A METHOD OF GRADIENT TYPE WITH RESPECT TO THE PRIMAL AND DUAL VARIABLES

Let us rewrite system (5.4) once more in terms of the Lagrange function

$$\mathcal{L}(v, w, p) = \langle F(v), w - v\rangle + \langle p, g(v, w)\rangle$$

for all $w \in \Omega_0$, $\forall p \geq 0$, and $v \in \Omega_0$, i.e.,

$$v^* = \pi_{\Omega_0}(v^* - \alpha\nabla_w\mathcal{L}(v^*, v^*, p^*)), \qquad p^* = \pi_+(p^* + \alpha g(v^*, v^*)). \tag{6.1}$$

The discrepancy, that is, the difference between the left- and right-hand sides of Eq. (6.1), which vanishes at the point (v^*, p^*) and is in general nonzero at an arbitrary point (v, p), determines a transformation of the space $R^n \times R^n$ into itself. The image of this transformation can be considered as a vector field [20] whose fixed point is (v^*, p^*). We consider the problem of drawing a trajectory such that the velocity vector coincides with the given direction of the field at this point. This problem is formally described by the system of differential equations

$$\frac{dv}{dt} + v = \pi_{\Omega_0}\{v - \alpha\nabla_w\mathcal{L}(v, v, p)\}, \quad \frac{dp}{dt} + p = \pi_+\{p + \alpha g(v, v)\}, \quad v(t_0) = v^0, \;\; p(t_0) = p^0,$$

where $\alpha > 0$ is an increment-like parameter. To provide the convergence of this trajectory to a saddle point of the Lagrange function $\mathcal{L}(v^*, w, p)$ for a given value $v = v^*$ of problem (1.1), we introduce an additive feedback control. The choice of various types of feedback results in various control differential systems. This approach was described in detail in [21, 22]. The general methodology of feedbacks was presented in the monograph [23].

In the present paper, we consider controlled processes of the form

$$\frac{dv}{dt} + v = \pi_{\Omega_0}\{v - \alpha\nabla_w\mathcal{L}(\bar{v}, \bar{v}, \bar{p})\}, \qquad \frac{dp}{dt} + p = \pi_+\{p + \alpha g(\bar{v}, \bar{v})\}, \tag{6.2}$$

where the controls have the form $\bar{v} = \pi_{\Omega_0}\{v - \alpha\nabla_w\mathcal{L}(v, v, \bar{p})\}$ and $\bar{p} = \pi_+\{p + \alpha g(v, v)\}$. An iterative analog of this system is

$$\begin{array}{rclrcl} \bar{p}^n &=& \pi_+(p^n + \alpha g(v^n, v^n)), & \qquad \bar{v}^n &=& \pi_{\Omega_0}(v^n - \alpha\nabla_w\mathcal{L}(v^n, v^n, \bar{p}^n)), \\ p^{n+1} &=& \pi_+(p^n + \alpha g(\bar{v}^n, \bar{v}^n)), & \qquad v^{n+1} &=& \pi_{\Omega_0}(v^n - \alpha\nabla_w\mathcal{L}(\bar{v}^n, \bar{v}^n, \bar{p}^n)). \end{array}$$

The increment $\alpha > 0$ in the process (6.2) is found from the condition $0 < \alpha < \alpha_0$, where α_0 is a constant to be specified below. We also suppose that the Lipschitz necessary conditions

$$|g(v + h, v + h) - g(v, v)| \leq |g|\,|h| \tag{6.3}$$

are satisfied for all $v \in \Omega$ and $h \in R^n$, where $|g|$ is a constant; $|F(v + h) - F(v)| \leq |F|\,|h|$ and $|\nabla_w^\top g(v + h, v + h) - \nabla_w^\top g(v, v))| \leq |\nabla|\,|h|$ for all $v \in \Omega$ Ë $h \in R^n$, where $|F|$, $|\nabla|$ are constants. In addition, we assume that $|\bar{p}^n| \leq C_0$ for all $n \to \infty$.

With regard for these conditions, it follows from (6.2) that the estimates

$$|\dot{p} + p - \bar{p}| \leq \alpha|g(\bar{v}, \bar{v}) - g(v, v)| \leq \alpha|g|\,|\bar{v} - v|, \tag{6.4}$$

$$
\begin{aligned}
|\dot{v} + v - \bar{v}| &\le \alpha|\nabla_w \mathcal{L}(\bar{v}, \bar{v}, \bar{p}) - \nabla_w \mathcal{L}(v, v, \bar{p})| \le \\
&\le \alpha|F(\bar{v}) + \nabla_w^\top g(\bar{v}, \bar{v})\bar{p} - F(v) - \nabla_w^\top g(v, v)\bar{p})| \le \\
&\le \alpha(|F| + C_0|\nabla|)|\bar{v} - v| = \alpha C|\bar{v} - v|,
\end{aligned} \tag{6.5}
$$

hold, where $C = |F| + C_0|\nabla|$.

Let us represent system (6.2) in the form of the variational inequalities

$$\langle \dot{v} + \alpha\nabla_w \mathcal{L}(\bar{v}, \bar{v}, \bar{p}), w - v - \dot{v}\rangle \ge 0, \qquad \forall w \in \Omega_0, \tag{6.6}$$

$$\langle \dot{p} - \alpha g(\bar{v}, \bar{v}), y - p - \dot{p}\rangle \ge 0, \qquad \forall y \ge 0, \tag{6.7}$$

$$\langle \bar{v} - v + \alpha\nabla_w \mathcal{L}(v, v, \bar{p}), w - \bar{v}\rangle \ge 0, \qquad \forall w \in \Omega_0, \tag{6.8}$$

$$\langle \bar{p} - p - \alpha g(v, v), y - \bar{p}\rangle \ge 0, \qquad \forall y \ge 0. \tag{6.9}$$

We claim that each trajectory of the process (6.2) monotonically converges in norm to some equilibrium solution.

Theorem 1. *Suppose that the set of solutions of problem* (1.1) *is nonempty, $F(v)$ is a monotone operator, the vector function $g(v, w)$ is symmetric and convex in w for each v, $|p(t)| \le C$ for all $t \to \infty$, the restriction $g(v, w)|_{v=w}$ of g to the diagonal of the square is a convex function, and $\Omega \subseteq I\!R^n$ is a convex closed set. Then the trajectory $v(t)$, $p(t)$ of the process* (6.2) *with $0 < \alpha < \alpha_0$ monotonically converges in norm to some equilibrium solution, i.e., $v(t) \to v^*$ and $p(t) \to p^*$ as $t \to \infty$.*

Proof. Setting $w = v^* \in \Omega^*$ in (6.6), we obtain

$$\langle \dot{v} + \alpha(F(\bar{v}) + \nabla_w^\top g(\bar{v}, \bar{v})\bar{p}), v^* - v - \dot{v}\rangle \ge 0. \tag{6.10}$$

Let $w = v + \dot{v}$; then $\langle \bar{v} - v + \alpha(F(v) + \nabla_w^\top g(v, v)\bar{p}), v + \dot{v} - \bar{v}\rangle \ge 0$. Hence

$$
\begin{aligned}
\langle \bar{v} - v, v + \dot{v} - \bar{v}\rangle &+ \alpha\langle F(\bar{v}), v + \dot{v} - \bar{v}\rangle - \alpha\langle F(\bar{v}) - F(v), v + \dot{v} - \bar{v}\rangle + \\
&+ \alpha\langle \nabla_w^\top g(\bar{v}, \bar{v})\bar{p}, v + \dot{v} - \bar{v}\rangle - \\
&- \alpha\langle (\nabla_w^\top g(\bar{v}, \bar{v}) - \nabla_w^\top g(v, v))\bar{p}, v + \dot{v} - \bar{v}\rangle \ge 0,
\end{aligned}
$$

or, by (6.5),

$$\langle \bar{v} - v, v + \dot{v} - \bar{v}\rangle + \alpha\langle F(\bar{v}), v + \dot{v} - \bar{v}\rangle + \alpha\langle \nabla_w^\top g(\bar{v}, \bar{v})\bar{p}, v + \dot{v} - \bar{v}\rangle + (\alpha C)^2|\bar{v} - v|^2 \ge 0. \tag{6.11}$$

By summing inequalities (6.10) and (6.11), we obtain

$$
\begin{aligned}
\langle \dot{v}, v^* - v - \dot{v}\rangle &+ \langle \bar{v} - v, v + \dot{v} - \bar{v}\rangle + \alpha\langle F(\bar{v}), v^* - \bar{v}\rangle + \\
&+ \alpha\langle \bar{p}, \nabla_w g(\bar{v}, \bar{v})(v^* - \bar{v})\rangle + (\alpha C)^2|\bar{v} - v|^2 \ge 0.
\end{aligned} \tag{6.12}
$$

Taking into account relation (3.6) and the componentwise convexity of the function $g(v, v)$, we separately transform the fourth term in (6.12):

$$\langle \bar{p}, \nabla_w g(\bar{v}, \bar{v})(v^* - \bar{v})\rangle \le 0.5\langle \bar{p}, g(v^*, v^*) - g(\bar{v}, \bar{v})\rangle,$$

Then inequality (6.12) acquires the form

$$
\begin{aligned}
\langle \dot{v}, v^* - v - \dot{v}\rangle &+ \langle \bar{v} - v, v + \dot{v} - \bar{v}\rangle + \alpha\langle F(\bar{v}), v^* - \bar{v}\rangle + \\
&+ (\alpha/2)\langle \bar{p}, g(v^*, v^*) - g(\bar{v}, \bar{v})\rangle + (\alpha`)^2|\bar{v} - v|^2 \ge 0.
\end{aligned} \tag{6.13}
$$

Setting $w = \bar{v}$ in the first inequality in (5.5) and adding the resulting inequality to (6.13), we obtain

$$\begin{aligned} \langle \dot{v}, v^* - v - \dot{v} \rangle &+ \langle \bar{v} - v, v + \dot{v} - \bar{v} \rangle + \alpha \langle F(\bar{v}) - F(v^*), v^* - \bar{v} \rangle + \\ &+ (\alpha/2) \langle \bar{p} - p^*, g(v^*, v^*) - g(\bar{v}, \bar{v}) \rangle + (\alpha C)^2 |\bar{v} - v|^2 \geq 0. \end{aligned} \tag{6.14}$$

Let us consider inequalities (6.7) and (6.9). Setting $y = p^*$ in (6.7) and $y = p + \dot{p}$ in (6.9), we obtain

$$\langle \dot{p}, p^* - p - \dot{p} \rangle - \alpha \langle g(\bar{v}, \bar{v}), p^* - p - \dot{v} \rangle \geq 0, \tag{6.15}$$

$$\langle \bar{p} - p, p + \dot{p} - \bar{p} \rangle + \alpha \langle g(\bar{v}, \bar{v}) - g(v, v), p + \dot{p} - \bar{p} \rangle - \alpha \langle g(\bar{v}, \bar{v}), p + \dot{p} - \bar{p} \rangle \geq 0, \tag{6.16}$$

respectively. We estimate the second term in (6.16) with the use of (6.3) and (6.4). Then we sum inequalities (6.15) and (6.16):

$$\langle \dot{p}, p^* - p - \dot{p} \rangle + \langle \bar{p} - p, p + \dot{p} - \bar{p} \rangle + (\alpha |g|)^2 |\bar{v} - v|^2 - \alpha \langle g(\bar{v}, \bar{v}), p^* - \bar{p} \rangle \geq 0.$$

Using the relations $\langle \bar{p}, g(v^*, v^*) \rangle \leq 0$ and $\langle p^*, g(v^*, v^*) \rangle = 0$, we rewrite the last inequality as

$$\begin{aligned} 0.5 \langle \dot{p}, p^* - p - \dot{p} \rangle &+ 0.5 \langle \bar{p} - p, p + \dot{p} - \bar{p} \rangle + 0.5(\alpha^2/2) |g|^2 |\bar{v} - v|^2 + \\ &+ (\alpha/2) \langle g(v^*, v^*) - g(\bar{v}, \bar{v}), p^* - \bar{p} \rangle \geq 0. \end{aligned} \tag{6.17}$$

We sum inequalities (6.14) and (6.17) and take into account the monotonicity of the operator $F(v)$:

$$\begin{aligned} \langle \dot{v}, v^* - v - \dot{v} \rangle &+ \langle \bar{v} - v, v + \dot{v} - \bar{v} \rangle + 0.5 \langle \dot{p}, p^* - p - \dot{p} \rangle + \\ &+ 0.5 \langle \bar{p} - p, p + \dot{p} - \bar{p} \rangle + \alpha^2 (C^2 + 0.5 |g|^2) |\bar{v} - v|^2 \geq 0. \end{aligned}$$

We represent the resulting inequality in the form

$$\begin{aligned} \langle \dot{v}, v^* - v \rangle &+ 0.5 \langle \dot{p}, p^* - p \rangle - |\dot{v}|^2 - 0.5 |\dot{p}|^2 + \alpha^2 ({}^{\text{‘}2} + 0.5 |g|^2) |\bar{v} - v|^2 + \\ &+ \langle \bar{u} - v, v + \dot{v} - \bar{u} \rangle + \langle \bar{p} - p, p + \dot{p} - \bar{p} \rangle \geq 0. \end{aligned} \tag{6.18}$$

The last two inner products on the left-hand side in (6.18) can be transformed with the use of the identities

$$\begin{aligned} 2 \langle \bar{u} - v, v + \dot{v} - \bar{u} \rangle &= |\dot{v}|^2 - |\bar{u} - v|^2 - |\bar{u} - v - \dot{v}|^2, \\ 2 \langle \bar{p} - p, p + \dot{p} - \bar{p} \rangle &= |\dot{p}|^2 - |\bar{p} - p|^2 - |\bar{p} - p - \dot{p}|^2. \end{aligned}$$

Then the inequality in question acquires the form

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |v - v^*|^2 &+ \frac{1}{2} |\dot{v}|^2 + \frac{1}{4} \frac{d}{dt} |p - p^*|^2 + \left(\frac{1}{2} - \alpha^2 \left(C^2 + \frac{1}{2} |g|^2 \right) \right) |\bar{v} - v|^2 + \\ &+ \frac{1}{2} |\bar{p} - p|^2 + \frac{1}{2} |\bar{p} - p - \dot{p}|^2 \leq 0. \end{aligned}$$

Let $\alpha_0 = 0.5 - \alpha^2 (C^2 + 0.5 |g|^2)$; then we integrate the resultant inequality from t_0 to t:

$$\begin{aligned} |v - v^*|^2 + \frac{1}{2} |p - p^*|^2 + \int_{t_0}^{t} |\dot{v}|^2 d\tau + + 2\alpha_0 \int_{t_0}^{t} |\bar{v} - v|^2 d\tau + \int_{t_0}^{t} |\bar{p} - p|^2 d\tau + \int_{t_0}^{t} |\bar{p} - p - \dot{p}|^2 d\tau \leq \\ \leq |v_0 - v^*|^2 + 0.5 |p_0 - p^*|^2. \end{aligned}$$

Hence the quantity $|v(t)-v^*|+0.5|p(t)-p^*|$ monotonically decays, $v(t)$, $p(t)$ is a bounded trajectory, and, moreover, the following integrals are convergent:

$$\int_{t_0}^{t} |\dot{v}|^2 d\tau < \infty, \quad \int_{t_0}^{t} |\bar{v}-v|^2 d\tau < \infty, \quad \int_{t_0}^{t} |\bar{p}-p|^2 d\tau < \infty, \quad \int_{t_0}^{t} |\bar{p}-p-\dot{p}|^2 d\tau < \infty$$

as $t \to \infty$. Let us prove the convergence of the trajectory $(v(t), p(t))$ to an equilibrium solution of the problem. Supposing that there exists an $\varepsilon > 0$ such that $|\dot{v}(t)| \geq \varepsilon$, $|\dot{p}(t)| \geq \varepsilon$, $|v-\bar{v}|^2 \geq \varepsilon$ and $|p-\bar{p}|^2 \geq \varepsilon$ for all $t \geq t_0$, we arrive at a contradiction with the convergence of the integrals. Consequently, there exists a subsequence $t_i \to \infty$ such that $|\dot{v}(t_i)| \to 0$, $|\dot{p}(t_i)| \to 0$, and $|v(t_i)-\bar{v}(t_i)| \to 0$. Since $(v(t), p(t))$ is bounded, we once more choose a subsequence of instants (we denote it again by t_i) such that $|v(t_i)| \to v'$, $|p(t_i)| \to p'$, $|v(t_i)-\bar{v}(t_i)| \to 0$, $|\dot{v}(t_i)| \to 0$, and $|\dot{p}(t_i)| \to 0$. Let us consider Eq. (6.2) for all $t_i \to \infty$. Passing to the limit, we obtain the relations $v' = \pi_{\Omega_0}(v' - \alpha\nabla_w \mathcal{L}(v', v', p'))$ and $p' = \pi_+(p' - \alpha g(v', v'))$. They coincide with (6.1); consequently, $v' = v^* \in \Omega^*$ and $p' = p^* \geq 0$. Therefore, any limit point of the trajectory $v(t)$ is a solution of problem (1.1). Since $|v(t)-v^*| + 0.5|p(t)-p^*|$ monotonically decays, it follows that $(v(t), p(t))$ has a unique limit point. The proof of the theorem is complete. □

The above can be generalized if we consider the method in the presence of noises.

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