PERGAMON Nonlinear Analysis 47 (2001) 1833–1844

Differential Equations for Equilibrium Problems with Coupled Constraints

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Revised 27 November 2003

Abstract. The equilibrium programming problem with coupled constraints is presented. A properties of symmetric and skew-symmetric functions are discussed. New concept of symmetric coupled constraints is offered. The differential feed-back control gradient-type method for solving the coupled constrained equilibrium problem is suggested and its convergence is proved.

1 Statement of problem

Let us consider the problem of computing a fixed point of a extreme coupled constrained mapping: find $v^* \in \Omega_0$ such that

$$v^* \in \text{Argmin}\{\Phi(v^*, w) \mid g(v^*, w) \leq 0, \quad w \in \Omega_0\}, \tag{1.1}$$

where $\Phi(v,w) : R^n \times R^n \to R$, $g(v,w) : R^n \times R^n \to R^m$, $\Omega_0 \in R^n$ is a convex closed set. It is assumed that $\Phi(v,w)$ and each component of vector-function $g(v,w)$ are convex in $w \in \Omega_0$ for any $v \in \Omega_0$. It is also assumed that the extreme (marginal) mapping $w(v) \equiv \text{Argmin}\{\Phi(v,w) \mid g(v,w) \leq 0,\ w \in \Omega_0\}$ is defined for all $v \in \Omega_0$ and the solution set $\Omega^* = \{v^* \in \Omega \mid v^* \in w(v^*)\} \subset \Omega_0$ of the initial problem is non-empty. According [1] this approval follows from the continuity of $\Phi(v,w)$ and the convexity of $\Phi(v,w)$ in w for any $v \in \Omega_0$, where Ω_0 is compact.

By definition of (1.1), any fixed point satisfies the inequality

$$\Phi(v^*, v^*) \leq \Phi(v^*, w), \qquad g(v^*, w) \leq 0 \ \forall w \in \Omega_0. \tag{1.2}$$

Let us introduce the function $\Psi(v,w) = \Phi(v,w) - \Phi(v,v)$ and using it we present (1.2) as

$$\Psi(v^*, w) \geq 0, \quad g(v^*, w) \leq 0 \qquad \forall w \in \Omega_0. \tag{1.3}$$

The inequality obtained means that second component of point v^*, v^* is a minimum of $\Psi(v^*, w)$ in w over "vertical" section $S_1(w) = \{w \mid g(v^*, w) \leq 0, w \in \Omega_0\}$. On the other hand it can occur so that the first component of this point is also a maximum of $\Psi(v, v^*)$ over "horizontal" section $S_2(v) = \{v \mid g(v, v^*) \leq 0, v \in \Omega_0\}$, i.e.

$$\Psi(v, v^*) \leq 0, \quad g(v, v^*) \leq 0 \qquad \forall w \in \Omega_0. \tag{1.4}$$

Problem (1.4) we call hereinafter as dual. Combining conditions (1.3) and (1.4), we see that pair v^*, v^* is a saddle point of function $\Psi(v,w)$, i.e.

$$\Psi(v,v^*) \le \Psi(v^*,v^*) \le \Psi(v^*,w), \quad g(v,v^*) \le 0, \quad g(v^*,w) \le 0 \qquad \forall v,w \in \Omega_0. \quad (1.5)$$

Certainly, in general case the pair v^*, v^* is not always saddle point but in many practically significant situations it is so. Moreover, non-saddle pair v^*, v^* it is possible frequently to result to a saddle-point situation with the help of procedures of splitting of bi-functions (functions of two variables) in symmetric and anti-symmetric parts. The procedure of decomposition of functions is considered in the consequent sections.

The equilibrium problem (1.1) can be considered as scalarization or convolution for many $n-$person game with coupled constraints [2]. For the first time the n-person game on square was considered in [3]. The scalarization procedure for game without coupled constraints was described in [4].

2 Splitting of functions

In the linear space of scalar or vector bifunctions $F(v,w)$ we mark out two linear subspaces by means of conditions

$$F(v,w) - F(w,v) \;=\; 0 \quad \forall w \in \Omega_0,\ \forall v \in \Omega_0, \quad (2.1)$$
$$F(v,w) + F(w,v) \;=\; 0 \quad \forall w \in \Omega_0,\ \forall v \in \Omega_0. \quad (2.2)$$

The functions of the first subspace are called symmetric; those of the second class, anti-symmetric. If these functions are defined on a square grid, we have the conventional classes of symmetric and anti-symmetric matrices.

Recall that a pair of points with coordinates w, v and v, w is situated symmetrically concerning the diagonal of the square $\Omega_0 \times \Omega_0$, i.e., with respect to the linear manifold $v = w$. This allows us to introduce the concept of a transposed function [5]. If we assign the value of $\Phi(w,v)$ calculated at the point w, v to every point with coordinates v, w, that is $v, w \to F(w,v)$, then we obtain the transposed function $F^\top(v,w) = F(w,v)$. In terms of this function conditions (2.1) and (2.2) look like $F(v,w) = F^\top(v,w)$, $F(v,w) = -F^\top(v,w)$. Using the obvious relations $F(v,w) = (F^\top(v,w))^\top$, $(F_1(v,w)+F_2(v,w))^\top = F_1^\top(v,w) + F_2^\top(v,w)$, we can readily verify that any real function $F(v,w)$ can be represented as the sum

$$F(v,w) = S(v,w) + K(v,w), \quad (2.3)$$

where $S(v,w)$ and $K(v,w)$ are symmetric and anti-symmetric functions, respectively. This expansion is unique, and

$$S(v,w) = \frac{1}{2}(F(v,w) + F^\top(v,w)), \qquad K(v,w) = \frac{1}{2}(F(v,w) - F^\top(v,w)). \quad (2.4)$$

The classes of symmetric and anti-symmetric functions are subsets for a more expansive functional classes, namely, of pseudo-symmetric and skew-symmetric functions [6]. In the following section we will investigate properties of classes for these functions.

3 Symmetric and skew-symmetric functions

Consider some properties of symmetric functions. First of all we note that the set $M = \{v, w \mid g(v,w) \le 0,\ v,w \in \Omega_0 \times \Omega_0\}$ of problem (1.1) induced by symmetric vectorial function $g(v,w)$ is symmetric with regard to diagonal of square $\Omega_0 \times \Omega_0$, i.e. any two points with coordinates v, w and w, v always belong or do not belong to set M. This property easily follows from a condition of symmetry

$$g(v,w) = g(w,v) \quad \forall w \in \Omega_0,\ \forall v \in \Omega_0. \tag{3.1}$$

It is not hard to produce examples of symmetric functions. First of all they are functions generating budget constraints in economic equilibrium models: $g(v,w) = \langle v, w\rangle$ or $g(v,w) = \langle Av, w\rangle$, where A is a symmetric matrix. In applications the Cobb–Douglas and constant elasticity-of-substitution production functions are widely known: $g(v,w) = Av^{\alpha}w^{\beta}$ and $g(v,w) = A(\alpha v^{-\Omega} + \beta w^{-\Omega})^{-\gamma/\Omega}$, where $A > 0$, $\alpha > 0$, $\beta > 0$, $\Omega > 0$ are parameters. If α and β are equal, then these functions are symmetric. Easy to check up that the function $\Phi(v,w) = f(x_1, y_2) + f(y_1, x_2)$, where $v = (y_1, y_2)$, $w = (x_1, x_2)$ is symmetric.

Explore the crucial properties of symmetric functions [6]. To this end we differentiate identity (3.1) in w, then

$$\nabla_w^{\top} g(v,w) = \nabla_v^{\top} g(w,v) \quad \forall w \in \Omega_0,\ \forall v \in \Omega_0, \tag{3.2}$$

where $\nabla_w^{\top} g(v,w)$, $\nabla_v^{\top} g(w,v)$ are $(m \times n)$-matrices, and $\nabla_v g_i(w,v)$, $\nabla_w g_i(v,w)$, $i = 1, 2, \dots, m$ are line-vectors.

Let's put $w = v$ in (3.2), then we have

$$\nabla_w^{\top} g(v,v) = \nabla_v^{\top} g(v,v) \quad \forall v \in \Omega_0. \tag{3.3}$$

Thus, we can formulate the following

Property 1 *The matrices of gradient-restrictions of vector symmetric functions $g(v,w)$ with respect to variable v and w onto the diagonal of the square $\Omega_0 \times \Omega_0$ are identical.*

By the definition of the differentiability for function $g(v,w)$ we get [7]

$$g(v+h, w+k) = g(v,w) + \nabla_v^{\top} g(v,w)h + \nabla_w^{\top} g(v,w)k + \Omega(v,w,h,k), \tag{3.4}$$

where $\Omega(v,w,h,k)/(|h|^2 + |k|^2)^{1/2} \to 0$ as $|h|^2 + |k|^2 \to 0$. Let $w = v$ and $h = k$ be, then using (3.3) we get from (3.4)

$$g(v+h, v+h) = g(v,v) + 2\nabla_w^{\top} g(v,v)h + \Omega(v,h), \tag{3.5}$$

where $\Omega(v,h)/|h| \to 0$ as $|h| \to 0$. Since (3.5) is a particular case of (3.4) it means that gradient-restriction $\nabla_w^{\top} g(v,w)|_{v=w}$ is the gradient $\nabla^{\top} g(v,v)$ of function $g(v,v)$, i.e.

$$2\nabla_w^{\top} g(v,w)|_{v=w} = \nabla^{\top} g(v,v) \quad \forall v \in \Omega_0. \tag{3.6}$$

Hence, it proves as follows [6]

Property 2 *The operator* $2\nabla_w g(v,w)|_{v=w}$ *is potential, i.e.* $2\nabla_w^\top g(v,v) = \nabla^\top g(v,v)$.

This key property play further important role.

We will expand the class of anti-symmetric functions to a class of positive-semidefinite (skew-symmetric) one and will show that the equilibrium solutions of (1.1) with skew-symmetric objective functions have saddle property (1.5).

Definition 1 *A function* $\Phi(v,w)$ *from* $I\!R^n \times I\!R^n$ *to* $I\!R^1$ *is called positive-semidefinite or skew-symmetric onto* $\Omega_0 \times \Omega_0$*, if it obeys the inequality*

$$\Phi(w,w) - \Phi(w,v) - \Phi(v,w) + \Phi(v,v) \geq 0 \quad \forall w \in \Omega_0,\ \forall v \in \Omega_0. \tag{3.7}$$

The class of skew-symmetric functions is not empty as it includes all anti-symmetric functions (2.2). If we put in (2.2) $F(v,w) = \Phi(v,w)$ and $v = w$, then we receive $\Phi(v,v)+$ $+\Phi(v,v) = 0$, i.e. the anti-symmetric function is identically equal to zero on diagonal of square $\Omega_0 \times \Omega_0$. If the anti-symmetric function is convex in w, then it follows from (2.2) that it is concave in v, i.e. in this case $\Phi(v,w)$ is saddle-point function.

As an example of the anti-symmetric function we specify the normalized function $\Phi(v,w) = L(z,p) - L(x,y)$, where $w = (z,y)$, $v = (x,p)$ for a saddle-point problem

$$L(x^*,y) \leq L(x^*,p^*) \leq L(z,p^*) \quad \forall z \in Q \subseteq R^n,\ \forall y \in P \subseteq R^m. \tag{3.8}$$

Here $x^*, p^* \in Q \times P$ is a saddle point of function $L(z,y)$. Easy to check up that the normalized function in this example satisfies condition (2.2) [8]. From above it follows that the skew-symmetric equilibrium problems largely inherit properties saddle-point problems and include the last.

Note that condition (3.7) in the case of monotonicity for $\Phi(v,w)$ in $w \in \Omega_0$ entails the monotonicity of gradient-restriction $\nabla_w \Phi(v,w)|_{v=w}$. Indeed, let function $\Phi(v,w)$ be convex in w, then using the system of convex inequalities

$$\langle \nabla f(x), y - x\rangle \leq f(y) - f(x) \leq \langle \nabla f(y), y - x\rangle \tag{3.9}$$

for all x and y over some set, from (3.7) we have the monotonicity of gradient-restriction

$$\langle \nabla_w \Phi(w,w) - \nabla_w \Phi(v,v), w - v\rangle \geq 0 \quad \forall v,w \in \Omega_0. \tag{3.10}$$

Note that if $\Phi(v,w) = L(z,p) - L(x,y)$ is taking from (3.8), then $(-\nabla_x L(x,y), \nabla_y L(x,y))^\top$ is monotone operator. The latter fact was established in [9].

4 Reduction to double saddle point

We show now that if objective function $\Phi(v,w)$ in (1.1) is skew-symmetric, and the functional constraints are induced symmetric vectorial function $g(v,w)$, then a solution of this problem is subjected to saddle-point property (1.5). Really, using (1.2) from (3.7), if $v = v^*$, and symmetry of set $M = \{v, w \mid g(v,w) \leq 0, v, w \in \Omega_0 \times \Omega_0\}$, we have

$$\Phi(v,v) - \Phi(v,v^*) \geq 0, \quad g(v,v^*) \leq 0, \quad \forall v \in \Omega_0, \tag{4.1}$$

or

$$\Psi(v,v^*) \le 0, \quad g(v,v^*) \le 0 \quad \forall v \in \Omega_0. \tag{4.2}$$

Combining inequality (1.3) and (4.2), we can write

$$\Psi(v,v^*) \le \Psi(v^*,v^*) \le \Psi(v^*,w), \qquad g(v,v^*) \le 0, \quad g(v^*,w) \le 0 \qquad \forall v,w \in \Omega_0. \tag{4.3}$$

Obviously, this inequality coincides with (1.5). Thus, the equilibrium solution of (1.1) with skew-symmetric objective function and symmetric functional constraints satisfies saddle-point property (1.5).

If a function $\Psi(v,w)$ or $\Phi(v.w)$ is differentiable in w for any v from (1.2) we have

$$\langle \nabla_w \Phi(v^*,v^*), w - v^* \rangle \ge 0 \quad \forall w \in \Omega_0, \;\; g(v^*,w) \le 0. \tag{4.4}$$

Comparing the monotonicity condition (3.10) in $v = v^*$ and (4.4), we receive

$$\langle \nabla_w \Phi(v,v), v - v^* \rangle \ge 0 \quad \forall v \in \Omega_0, \;\; g(v,v^*) \le 0. \tag{4.5}$$

We record inequalities (4.4) and (4.5) as the system

$$\begin{gathered} \langle \nabla_w \Phi(v,v), v^* - v \rangle \le \langle \nabla_w \Phi(v^*,v^*), v^* - v^* \rangle \le \langle \nabla_w \Phi(v^*,v^*), w - v^* \rangle, \\ g(v,v^*) \le 0, \quad g(v^*,w) \le 0 \qquad \forall v,w \in \Omega_0. \end{gathered} \tag{4.6}$$

Thus, the pair v^*, v^* is a saddle point for function $\langle \nabla_w \Phi(v,v), w - v \rangle$ over the symmetric set $M = \{v,w \mid g(v,w) \le 0, \;\; v,w \in \Omega_0 \times \Omega_0\}$.

In optimization, the functional constraints are taken into account usually with the help of Lagrange functions. Obviously, this approach is useful in equilibrium problems too. To this end we consider analogs Lagrange functions for equilibrium programming problems

$$\mathcal{L}(v,w,p) = \Psi(v,w) + \langle p, g(v,w) \rangle \quad \forall v \in \Omega_0, \;\; w \in \Omega_0, \;\; p \ge 0,$$

and

$$\mathcal{L}_1(v,w,p) = \langle \nabla_w \Phi(v,v), w - v \rangle + \langle p, g(v,w) \rangle \quad \forall v \in \Omega_0, \;\; w \in \Omega_0, \;\; p \ge 0,$$

where $\Psi(v,w) = \Phi(v,w) - \Phi(v,v)$. If admissible set $\{w \mid g(v^*,w) \le 0, w \in \Omega_0\}$ at $v = v^*$ satisfies a regularity condition, for example, Slater condition, then problem (1.1) represents itself a convex programming problem with respect to variable w, and its Lagrange function, for example, $\mathcal{L}_1(v^*,w,p)$ has a saddle point v^*, p^*, which is subjected to the system of inequalities

$$\mathcal{L}_1(v^*,v^*,p) \le \mathcal{L}_1(v^*,v^*,p^*) \le \mathcal{L}_1(v^*,w,p^*) \quad \forall w \ge \Omega_0, \;\; \forall p \ge 0. \tag{4.7}$$

In the case of differentiability of function $g(v,w)$ this system of inequalities can be presented in the form of variational inequalities

$$\begin{gathered} \langle \nabla_w \Phi(v^*,v^*) + \nabla_w^\top g(v^*,v^*) p^*, w - v^* \rangle \ge 0 \quad \forall w \in \Omega_0, \\ \langle p - p^*, g(v^*,v^*) \rangle \le 0 \quad \forall p \ge 0. \end{gathered} \tag{4.8}$$

We transform separately second term in the first inequality (4.8). Taking into account the key property of symmetric functions (3.6) and convexity of vectorial function $g(v,v)$ componently, we have

$$\langle \nabla_w^\top g(v^*,v^*) p^*, w - v^* \rangle = \frac{1}{2} \langle p^*, \nabla g(v^*,v^*)(w - v^*) \rangle \le \frac{1}{2} \langle p^*, g(w,w) - g(v^*,v^*) \rangle \ge 0.$$

In view of an obtained evaluation we copy the first inequality from (4.8) in the form

$$\langle \nabla_w \Phi(v^*, v^*), w - v^* \rangle + \frac{1}{2} \langle p^*, g(w, w) - g(v^*, v^*) \rangle \geq 0 \quad \forall w \in \Omega_0. \tag{4.9}$$

If the operator $\nabla_w \Phi(v, v)$ is monotone, then by virtue of (3.10) we get from (4.9)

$$\langle \nabla_w \Phi(w, w), w - v^* \rangle + \frac{1}{2} \langle p^*, g(w, w) - g(v^*, v^*) \rangle \geq 0 \quad \forall w \in \Omega_0. \tag{4.10}$$

The first inequality from (4.8) and (4.10) on the whole are analog of (4.6) without functional constraints already. The inequality (4.10) is obtained from the monotonicity condition of operator $\nabla_w \Phi(v, v)$. However this inequality can be valid for the non-monotone operators. It is key property and is underlying the convergence analysis of gradient-type methods to the equilibrium solutions.

5 Gradient prediction-type method

System of inequalities (4.7) or (4.8) can be presented by means of projection operators in the form

$$v^* = \pi_{\Omega_0}(v^* - \alpha \nabla_w \mathcal{L}_1(v^*, v^*, p^*)), \qquad p^* = \pi_+(p^* + \alpha g(v^*, v^*)), \tag{5.1}$$

where $\pi_+(\ldots)$, $\pi_{\Omega_0}(\ldots)$ are the projection operators of some vector into the positive orthant R^n_+, Ω_0 is a convex set, $\alpha > 0$ is a parameter like steplength. We note that systems (4.7), (4.8), and (5.1) are equivalent.

Discrepancy, i.e. residual between left-hand and right-hand sides of (5.1), equal to zero at the point v^*, p^* and not equal to zero in any point v, p puts a transformation of space $R^n \times R^n$ in itself. The image of this transformation can be considered as vectorial field, where v^*, p^* is the fixed point. We shall deliver a problem about realization of trajectory such that a velocity vector of trajectory coincides with a specific direction of vector field at given point. The formal problem is described by the following system of differential equations of the form

$$\frac{dv}{dt} + v = \pi_{\Omega_0}\{v - \alpha \nabla_w \mathcal{L}_1(v, v, p)\},$$

$$\frac{dp}{dt} + p = \pi_+\{p + \alpha g(v, v)\}, \quad v(t_0) = v^0, \;\; p(t_0) = p^0,$$

where $\alpha > 0$. To supply convergence of this trajectory to a saddle point of the Lagrange function $\mathcal{L}_1(v^*, w, p)$ at $v = v^*$ of (1.1) we introduce a additive control in the feed-back form. Selection of various types of feed-backs control results in various controlled differential systems. This technique is rather detailed described in paper [10].

In this article we consider controlled processes of a kind

$$\frac{dv}{dt} + v = \pi_{\Omega_0}\{v - \alpha \nabla_w \mathcal{L}_1(\bar{v}, \bar{v}, \bar{p})\}, \qquad \frac{dp}{dt} + p = \pi_+\{p + \alpha g(\bar{v}, \bar{v})\}, \tag{5.2}$$

where the controls look like

$$\bar{v} = \pi_{\Omega_0}\{v - \alpha \nabla_w \mathcal{L}_1(v, v, \bar{p})\}, \qquad \bar{p} = \pi_+\{p + \alpha g(v, v)\}.$$

The iterated analog of this system takes the form [2]

$$\begin{aligned}
\bar{p}^n &= \pi_+(p^n + \alpha g(v^n, v^n)),\\
\bar{v}^n &= \pi_{\Omega_0}(v^n - \alpha \nabla_w \mathcal{L}_1(v^n, v^n, \bar{p}^n)),\\
p^{n+1} &= \pi_+(p^n + \alpha g(\bar{v}^n, \bar{v}^n)),\\
v^{n+1} &= \pi_{\Omega_0}(v^n - \alpha \nabla_w \mathcal{L}_1(\bar{v}^n, \bar{v}^n, \bar{p}^n)).
\end{aligned}$$

The steplength $\alpha > 0$ in (5.2) is determined from the condition $0 < \alpha < \alpha_0$, where the constant α_0 will be estimated below. It is supposed also that Lipschitz conditions are fulfilled

$$|g(v+h, v+h) - g(v, v)| \le |g|\,|h| \tag{5.3}$$

for all $v \in \Omega$ and $h \in R^n$, where $|g|$ is a constant and

$$\begin{aligned}
|\nabla_w \Phi(v+h, v+h) - \nabla_w \Phi(v, v)| \le |\Phi|\,|h|,\\
|\nabla_w^\top g(v+h, v+h) - \nabla_w^\top g(v, v))| \le |\nabla|\,|h|,
\end{aligned} \tag{5.4}$$

for all $v \in \Omega$ and $h \in R^n$, where $|\Phi|$, $|\nabla|$ are constants, besides it is supposed also that $|\bar{p}^n| \le C_0$ for all $n \to \infty$. In view of entered conditions we have evaluations from (5.2)

$$|\dot{p} + p - \bar{p}| \le \alpha |g(\bar{v}, \bar{v}) - g(v, v)| \le \alpha |g|\,|\bar{v} - v|, \tag{5.5}$$

$$|\dot{v} + v - \bar{v}| \le \alpha |\nabla_w \mathcal{L}_1(\bar{v}, \bar{v}, \bar{p}) - \nabla_w \mathcal{L}_1(v, v, \bar{p})| \le \alpha(|\Phi| + C_0|\nabla|)|\bar{v} - v| = \alpha C |\bar{v} - v|, \tag{5.6}$$

where $C = |\Phi| + C_0|\nabla|$. We present the system of equations (5.2) as variational inequalities

$$\langle \dot{v} + \alpha \nabla_w \mathcal{L}_1(\bar{v}, \bar{v}, \bar{p}), w - v - \dot{v} \rangle \ge 0 \quad \forall w \in \Omega_0, \tag{5.7}$$

$$\langle \dot{p} - \alpha g(\bar{v}, \bar{v}), y - p - \dot{p} \rangle \ge 0 \quad \forall y \ge 0, \tag{5.8}$$

$$\langle \bar{v} - v + \alpha \nabla_w \mathcal{L}_1(v, v, \bar{p}), w - \bar{v} \rangle \ge 0 \quad \forall w \in \Omega_0, \tag{5.9}$$

$$\langle \bar{p} - p - \alpha g(v, v), y - \bar{p} \rangle \ge 0 \quad \forall y \ge 0. \tag{5.10}$$

We show that the trajectory of process (5.2) converges monotonically under the norm of space to one of equilibrium solutions.

Theorem 1 *Suppose that a solution set of (1.1) is non-empty, function $\Phi(v, w)$ is positive-semidefinite and convex in w for any v, vector-function $g(v, w)$ is symmetric, differentiable, and convex in w for any v, moreover its restriction $g(v, w)|_{v=w}$ on the diagonal of the square is convex function, $|p(t)| \le C$ for all $t \to \infty$, $\Omega \subseteq \mathbb{R}^n$ is convex closed set. Then, the trajectory $v(t), p(t)$ of (5.2) with the parameter $0 < \alpha < \alpha_0$ converges to a equilibrium solution, i.e. $v(t), p(t) \to v^*, p^*$ as $t \to \infty$ monotonically in the norm.*

Proof. Putting $w = v^* \in \Omega^*$ in (5.7), we have

$$\langle \dot{v} + \alpha(\nabla_w \Phi(\bar{v}, \bar{v}) + \nabla_w^\top g(\bar{v}, \bar{v})\bar{p}), v^* - v - \dot{v} \rangle \ge 0. \tag{5.11}$$

We put $w = v + \dot{v}$ in (5.9)

$$\begin{aligned}
&\langle \bar{v} - v, v + \dot{v} - \bar{v} \rangle + \alpha \langle \nabla_w \Phi(\bar{v}, \bar{v}), v + \dot{v} - \bar{v} \rangle - \alpha \langle \nabla_w \Phi(\bar{v}, \bar{v}) - \nabla_w \Phi(v, v), v + \dot{v} - \bar{v} \rangle +\\
&+ \alpha \langle \nabla_w^\top g(\bar{v}, \bar{v})\bar{p}, v + \dot{v} - \bar{v} \rangle - \alpha \langle (\nabla_w^\top g(\bar{v}, \bar{v}) - \nabla_w^\top g(v, v))\bar{p}, v + \dot{v} - \bar{v} \rangle \ge 0,
\end{aligned}$$

and take into account (5.6)

$$\begin{aligned}\langle \bar{v}-v, v+\dot{v}-\bar{v}\rangle + \alpha\langle \nabla_w \Phi(\bar{v},\bar{v}), v+\dot{v}-\bar{v}\rangle + \\ +\alpha\langle \nabla_w^\top g(\bar{v},\bar{v})\bar{p}, v+\dot{v}-\bar{v}\rangle + (\alpha C)^2|\bar{v}-v|^2 \geq 0.\end{aligned} \tag{5.12}$$

We combine both inequalities (5.11) and (5.12)

$$\begin{aligned}\langle \dot{v}, v^*-v-\dot{v}\rangle + \langle \bar{v}-v, v+\dot{v}-\bar{v}\rangle + \alpha\langle \nabla_w \Phi(\bar{v},\bar{v}), v^*-\bar{v}\rangle + \\ +\alpha\langle \bar{p}, \nabla_w g(\bar{v},\bar{v})(v^*-\bar{v})\rangle + (\alpha C)^2|\bar{v}-v|^2 \geq 0.\end{aligned} \tag{5.13}$$

Taking into account (3.6) and component-wise convexity of function $g(v,v)$, we transform fourth term from (5.13) separately

$$\langle \bar{p}, \nabla_w g(\bar{v},\bar{v})(v^*-\bar{v})\rangle = \frac{1}{2}\langle \bar{p}, \nabla g(\bar{v},\bar{v})(v^*-\bar{v})\rangle \leq \frac{1}{2}\langle \bar{p}, g(v^*,v^*)-g(\bar{v},\bar{v})\rangle,$$

then

$$\begin{aligned}\langle \dot{v}, v^*-v-\dot{v}\rangle + \langle \bar{v}-v, v+\dot{v}-\bar{v}\rangle + \alpha\langle \nabla_w \Phi(\bar{v},\bar{v}), v^*-\bar{v}\rangle + \\ +(\alpha/2)\langle \bar{p}, g(v^*,v^*)-g(\bar{v},\bar{v})\rangle + (\alpha')^2|\bar{v}-v|^2 \geq 0.\end{aligned}$$

We put $w=\bar{v}$ in (4.10), then

$$\langle \nabla_w \Phi(\bar{v},\bar{v}), \bar{v}-v^*\rangle + \frac{1}{2}\langle p^*, g(\bar{v},\bar{v})-g(v^*,v^*)\rangle \geq 0.$$

We combine two last inequalities

$$\langle \dot{v}, v^*-v-\dot{v}\rangle + \langle \bar{v}-v, v+\dot{v}-\bar{v}\rangle + \frac{\alpha}{2}\langle \bar{p}-p^*, g(v^*,v^*)-g(\bar{v},\bar{v})\rangle + (\alpha C)^2|\bar{v}-v|^2 \geq 0. \tag{5.14}$$

Consider inequalities (5.8) and (5.10). We set $p=p^*$ in (5.8)

$$\langle \dot{p}, p^*-p-\dot{p}\rangle - \alpha\langle g(\bar{v},\bar{v}), p^*-p-\dot{v}\rangle \geq 0,$$

and $y=p+\dot{p}$ in (5.10)

$$\langle \bar{p}-p, p+\dot{p}-\bar{p}\rangle + \alpha\langle g(\bar{v},\bar{v})-g(v,v), p+\dot{p}-\bar{p}\rangle - \alpha\langle g(\bar{v},\bar{v}), p+\dot{p}-\bar{p}\rangle \geq 0.$$

We estimate the second term in this inequality by means of (5.3) and (5.6), and add both inequalities

$$\langle \dot{p}, p^*-p-\dot{p}\rangle + \langle \bar{p}-p, p+\dot{p}-\bar{p}\rangle + (\alpha|g|)^2|\bar{v}-v|^2 - \alpha\langle g(\bar{v},\bar{v}), p^*-\bar{p}\rangle \geq 0.$$

Using relations $\langle \bar{p}, g(v^*,v^*)\rangle \leq 0$, $\langle p^*, g(v^*,v^*)\rangle = 0$, we rewrite the last inequality in the form

$$\frac{1}{2}\langle \dot{p}, p^*-p-\dot{p}\rangle + \frac{1}{2}\langle \bar{p}-p, p+\dot{p}-\bar{p}\rangle + \frac{\alpha^2}{2}|g|^2|\bar{v}-v|^2 + \frac{\alpha}{2}\langle g(v^*,v^*)-g(\bar{v},\bar{v}), p^*-\bar{p}\rangle \geq 0. \tag{5.15}$$

We combine inequalities (5.14) and (5.15)

$$\langle \dot{v}, v^*-v-\dot{v}\rangle + \langle \bar{v}-v, v+\dot{v}-\bar{v}\rangle + \frac{1}{2}\langle \dot{p}, p^*-p-\dot{p}\rangle + \frac{1}{2}\langle \bar{p}-p, p+\dot{p}-\bar{p}\rangle + \alpha^2\left(C^2+\frac{|g|^2}{2}\right)|\bar{v}-v|^2 \geq 0.$$

Present the inequality obtained in the kind

$$\begin{aligned}\langle \dot v, v^* - v\rangle &+ \frac{1}{2}\langle \dot p, p^* - p\rangle - |\dot v|^2 - \frac{1}{2}|\dot p|^2 + \alpha^2\left(C^2 + \frac{1}{2}|g|^2\right)|\bar v - v|^2 + \\ &+ \langle \bar u - v, v + \dot v - \bar u\rangle + \langle \bar p - p, p + \dot p - \bar p\rangle \geq 0.\end{aligned} \tag{5.16}$$

We transform two last scalar products in a left-hand part (5.15) with the help of identities

$$\begin{aligned}2\langle \bar u - v, v + \dot v - \bar u\rangle &= |\dot v|^2 - |\bar u - v|^2 - |\bar u - v - \dot v|^2, \\ 2\langle \bar p - p, p + \dot p - \bar p\rangle &= |\dot p|^2 - |\bar p - p|^2 - |\bar p - p - \dot p|^2.\end{aligned}$$

Then the considered inequality get the form

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v - v^*|^2 &+ \frac{1}{2}|\dot v|^2 + \frac{1}{4}\frac{d}{dt}|p - p^*|^2 + \left(\frac{1}{2} - \alpha^2\left(C^2 + \frac{1}{2}|g|^2\right)\right)|\bar v - v|^2 + \\ &+ \frac{1}{2}|\bar p - p|^2 + \frac{1}{2}|\bar p - p - \dot p|^2 \leq 0.\end{aligned}$$

If $\alpha_0 = (1/2) - \alpha^2(C^2 + (1/2)|g|^2) > 0$, then we integrate inequality obtained from t_0 up to t:

$$\begin{aligned}|v - v^*|^2 + \frac{1}{2}|p - p^*|^2 &+ \int_{t_0}^{t} |\dot v|^2 d\tau + 2\alpha_0 \int_{t_0}^{t} |\bar v - v|^2 d\tau + \int_{t_0}^{t} |\bar p - p|^2 d\tau + \int_{t_0}^{t} |\bar p - p - \dot p|^2 d\tau \leq \\ &\leq |v^0 - v^*|^2 + \frac{1}{2}|p^0 - p^*|^2.\end{aligned}$$

The monotone decrease of $|v(t) - v^*| + (1/2)|p(t) - p^*|$ follows from an evaluation obtained and boundedness of trajectory $v(t), p(t)$ we have as well, besides it follows convergence of all integrals. We prove convergence of trajectory $v(t), p(t)$ to the equilibrium solution of the problem. By admitting the existence $\varepsilon > 0$ such that $|\dot v(t)| \geq \varepsilon$, $|\dot p(t)| \geq \varepsilon$, $|v - \bar v|^2 \geq \varepsilon$, $|p - \bar p|^2 \geq \varepsilon$ for all $t \geq t_0$, we get the inconsistency to convergence of integrals. Therefore, there is the subsequence of times moment $t_i \to \infty$ such that $|\dot v(t_i)| \to 0$, $|\dot p(t_i)| \to 0$, $|v(t_i) - \bar v(t_i)| \to 0$. Since $v(t), p(t)$ is bounded, we select once again subsequence of times, which we also designate t_i, such that $|v(t_i)| \to v'$, $|p(t_i)| \to p'$, $|v(t_i) - \bar v(t_i)| \to 0$, $|\dot v(t_i)| \to 0$, $|\dot p(t_i)| \to 0$. We consider equation (5.2) for all time moment $t_i \to \infty$ and, passing into a limit, we write out marginal relations

$$v' = \pi_{\Omega_0}(v' - \alpha \nabla_w \mathcal{L}_1(v', v', p')), \qquad p' = \pi_+(p' - \alpha g(v', v')).$$

These equations coincide with (5.1) and, therefore, we have $v' = v^* \in \Omega^*$, $p' = p^* \geq 0$. Thus, any limit point of trajectory $v(t)$ is the solution of (1.1). By virtue of a monotonicity of decrease of size $|v(t) - v^*| + (1/2)|p(t) - p^*|$ trajectory $v(t), p(t)$ has get the unique limit point. The theorem is proved.

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