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Contents

Numerical Solution to an Inverse Coefficient Problem for Loaded Evolution Equations

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We consider a problem of restoration of the coefficients for a loaded one-dimensional parabolic equation [1]:

$$\frac{\partial u(x,t)}{\partial t} = \Im(x,t)u(x,t) + N(x,t)u(x,t) + F(x,t;\, C) + f(x,t),$$

$$(x,t) \in \Omega = (0;a) \times (0;T]. \tag{1}$$

Here $\Im(x,t)$ is the linear elliptic operator; $N(x,t)$ is the loading operator, with respect to which the following two forms are considered:

$$N(x,t)u(x,t) = \sum_{s=1}^{l_3} \widehat{b}_s(x,t)u(\widehat{x}_s,t), \tag{2}$$

$$N(x,t)u(x,t) = \sum_{s=1}^{l_3} \breve{b}_s(x,t)u(x,\breve{t}_s). \tag{3}$$

Here $u(x,t)$ is the phase state function; $\widehat{x}_s, \breve{t}_s$ are given loading points, $f(x,t)$, $\widehat{b}_s(x,t)$, $\breve{b}_s(x,t)$ are given functions continuous with respect to their arguments,$s = 1, 2, ..., l_3$.

Depending on which loading operator (2) or (3) is selected, the function $F(x,t;\, C)$ will have one of the following forms:

$$F(x,t;\, C) = \sum_{i=1}^{l} B_i(x,t)\, C_i(t), \tag{4}$$

$$F(x,t;\, C) = \sum_{i=1}^{l} B_i(x,t)\, C_i(x). \tag{5}$$

Here $B_i(x,t)$ are given continuous linearly independent functions; $C_i(x)$ and $C_i(t)$ are to be determined, $i = 1, 2, ..., l$.

To determine the identifiable functions $C_i(x)$ and $C_i(t)$, we have initial and boundary conditions given in the form of non-separated integral and point values of the phase state.

For example, in case of the loading operator (2), the initial, boundary, and auxiliary conditions will assume the following form:

$$u(x,0) = \varphi(x),\ 0 \le x \le a, \tag{6}$$

$$\sum_{i=1}^{l_1} \int_{\bar{x}_i}^{\bar{x}_i+\Delta_i} \bar{D}_i(x,t)u(x,t)dx + \sum_{j=1}^{l_2} \tilde{D}_j(t)u(\tilde{x}_j,t) + \\ + \sum_{s=1}^{l_3} \breve{D}_s(t)u(\breve{x}_s,t) = L_0(t),\ 0 \le t \le T, \tag{7}$$

where $\bar{t}_i,\ \tilde{t}_j,\ \breve{t}_s$ are given points of time belonging to$[0,T]$, $\bar{x}_i\ ,\quad \tilde{x}_j\ , \breve{x}_s \in [0;a]$, the function $\varphi(x)$, $(l+2)$-dimensional functions $\bar{D}_i(x,t)\ ,\quad \tilde{D}_j(t)\ ,$ $\breve{D}_s(t)$, $L_0(t)$ are given and continuous with respect to their arguments.

We propose an approach based on the application of the method of lines and reduction of the initial problem to a parametric identification problem for ordinary differential equations [2-3]. Next, we use a special representation of the solution to the derived boundary-value problem with respect to the linear system of differential equations with non-local conditions, with the help of which the parametric identification problem is reduced to the solution to auxiliary boundary-value problems and a system of algebraic equations.

The similar approach has been applied to the solution to inverse problems with respect to a loaded hyperbolic differential equation under non-local redefining conditions.

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Application of optimization methods for the solution of the material science problems

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Currently multyscale approaches are widely used for the purpose of predictable computational modeling of new materials [1]. At this approach the results of ab initio [2,3] modeling of atomic structure and electronic properties of cells containing 200-1000 atoms are used as input data for the modeling of more complex structures consisting of 1000-1000000 atoms. Such approaches provide wide possibilities for the modeling of structures with defects and allow to study various dynamical processes such as diffusion, adhesion, etc. Each scale of the model allows to formulate the problem at extremal setting and to apply relevant methods of optimization. At the level of the Density functional theory [1,2] the optimization problem of the finding of atomic positions corresponding to the ground state energy is solved. The electron density functional minimization is performed for each transitional atomic configuration.

In density functional theory, the total energy of a system is expressed as a functional of the charge density as:

$$E[\rho(\vec{r})] = F[\rho(\vec{r})] + \int U(\vec{r})\rho(\vec{r})d\vec{r}, \tag{1}$$

where $F[\rho(\vec{r})]$ is the functional not depending on the external potential, $U(\vec{r})$ is external potential. Kohn-Sham approach allows to write $F[\rho(\vec{r})]$ as follows:

$$F[\rho(\vec{r})] = T_s[\rho(\vec{r})] + \frac{1}{2}\int \frac{\rho(\vec{r})\rho(\vec{r'})}{|\vec{r}-\vec{r'}|}d\vec{r}d\vec{r'} + E_{xc}[\rho(\vec{r})], \tag{2}$$

where $T_s[\rho(\vec{r})]$ is kinetic energy functional, the second term describes Coulomb energy, and $E_{xc}[\rho(\vec{r})]$ is the exchange-correlation energy. Minimizing function is the self-consistent solution of the Schrdinger equation and equation for the external potential. Schrdinger equation:

$$\left(-\frac{1}{2}\nabla^2 + U_{eff}(\vec{r}) - \varepsilon_i\right)\phi_i(\vec{r}), \tag{3}$$

where $\phi_i(\vec{r})$ is single-particle wavefunction and $U_{eff}(\vec{r})$ is:

$$U_{eff}(\vec{r}) \equiv U(\vec{r}) + \int \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} d\vec{r'} + \nu_{xc}(\vec{r}), \tag{4}$$

where the second term is the Hartree potential, which describes Coulomb interaction of electrons, and the last term is the exchange-correlation potential:

$$\nu_{xc}(\vec{r}) \equiv \frac{\delta}{\delta\tilde{\rho}(\vec{r})} E_{xc}[\tilde{\rho}(\vec{r})]_{\tilde{\rho}(\vec{r})=\rho\vec{r}}, \tag{5}$$

Electron density $\rho\vec{r}$ can be written as follows:

$$\rho(\vec{r}) = \rho_s(\vec{r}) = \sum_{i=1}^{N} |\phi_i(\vec{r})|^2, \tag{6}$$

Equations (3), (4) and (6) are constituting the self-consistent Kohn-Sham equations[2]. This approach is implemented in VASP software. After the solution of the problem of the finding of the atomic structure of the ground state of the system and its structural and energetic parameters it is possible to make a transition to the next level of multyscale hierarchy, the molecular-dynamic modeling. On this level empirical potentials describing interatomic interactions are used. The structure of the interatomic potential depends on the type of the chemical bonding. The problem of the identifying of potential parameters for a given material is solved by optimization of the potential function [4]. Methods of local and global optimization are used [4].

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Implementation of Radically New Technologies in Leontief-type Models

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In real-world economies, apart from the evolutionary improvement of technologies and products, revolutionary new technologies are deployed and revolutionary new products are launched, which significantly affects the economy as a whole. Let us study ways of modeling the effects of launching radically new products. Accordingly, we will refer to this type of technological progress as *radical technological progress*. Suppose that the launch of a new product changes the old technology matrix $\check{A}$ of order n, whose elements are $\check{a}_{ij}$, for a new technology matrix $\hat{A}$ of order $n+1$, whose elements are $\hat{a}_{ij}$. Without loss of generality, we can assume that (1) the index of the new product and its corresponding technology is $n+1$ and that (2) the sector most affected by the new product is sector n, with product n being partially substituted with product $n+1$. More concretely, this substitution means that the new sector does not need the product of sector n such that we have $\hat{a}_{n(n+1)} = 0$. At the same time, we assume that the new technology matrix is irreducible. Since all the changes related to the launch are limited to the products whose indices are n and $n+1$, those elements of the technology matrix that are unrelated to these sectors remain unchanged: $\hat{a}_{ij} = \check{a}_{ij}$, $i = 1, \ldots, n-1$; $j = 1, \ldots, n$.

Now we introduce the following concept. Suppose an economic system has an irreducible technology matrix A (of order n), whose Frobenius vector is x. For this system, we consider the indicators $(a_{ij}x_j)/x_i$, $i, j = 1, \ldots, n$, which we will refer to as *relative unit cost of producing commodity j in terms of product i in balanced growth mode*.

Proposition 1. *The deployment of a radically new technology increases the von Neumann expansion rate if and only if the new technology decreases the sum, over all old sectors, of the relative unit costs in terms of the partially substituted product.*

Proof. Let us express the matrix $\hat{A}$ and its Frobenius vector$\hat{x}$ as $\hat{A} = \bar{A} + \delta A$, $\hat{x} = \bar{x} + \delta x$, where $\bar{A}$ is the square matrix of order $n+1$ whose column $n+1$ and row $n+1$ contain zero elements only and whose principle submatrix of order n obtained by removing column $n+1$ and

row $n+1$ equals $\check{A}$ and where $\bar{x}$ is the vector from $\mathbf{R}^{n+1}$ whose first n components form the Frobenius vector $\check{x}$ of the matrix $\check{A}$, and $\bar{x}_{n+1} = 0$. Here we assume that the vectors $\hat{x}$ and $\check{x}$ are such that $\hat{x}_n = \check{x}_n = 1$. We have

$$\hat{\lambda}\hat{x} = (\bar{A} + \delta A)(\bar{x} + \delta x) = \check{\lambda}\bar{x} + \bar{A}\delta x + \delta A(\bar{x} + \delta x) = \check{\lambda}\bar{x} + \bar{A}\delta x + \delta A\hat{x}.$$

If we consider component n of this vector equality only, we obtain

$$\hat{\lambda} = \check{\lambda} + \sum_{j=1}^{n+1} \hat{a}_{nj}\hat{x}_j - \sum_{j=1}^{n} \check{a}_{nj}\check{x}_j = \check{\lambda} + \sum_{j=1}^{n} \hat{a}_{nj}\hat{x}_j - \sum_{j=1}^{n} \check{a}_{nj}\check{x}_j.$$

This yields that $\hat{\lambda} < \check{\lambda}$ if and only if the sum of the unit costs in terms of n, over all old sectors, decreases. To complete the proof, recall that the von Neumann expansion rate of a model is indirectly proportional to the Frobenius eigenvalue of its technology matrix. □

We now consider the general case where the order of the technology matrix changes in an arbitrary way, but the matrix remains irreducible. Here we do not impose any constraints on the changes to the values of the input coefficients, except the requirement for them to be nonnegative.

Proposition 2. *The transition to a new technology matrix increases the von Neumann expansion rate if and only if the new technology matrix decreases the cost of producing one unit of gross output; the calculation of the cost and the gross output uses the same vector of equilibrium prices that correspond to the used technology matrix.*

Proof. Let $\check{p}$ and $\check{x}$ denote the left and right Frobenius vectors of the old technology matrix $\check{A}$, respectively. Suppose $\check{x}$ is a vector of outputs that belongs to the old von Neuman ray of the system. This vector of physical indicators is considered to be a unit of gross output if the following holds: $\langle \check{p}, \check{x} \rangle = 1$. For the new technology matrix $\hat{A}$, let $\hat{p}$ and $\hat{x}$ denote the analogous vectors, whose scalar product also equals one. If we multiply the equalities $\check{A}\check{x} = \check{\lambda}\check{x}$ and $\hat{A}\hat{x} = \hat{\lambda}\hat{x}$, by $\check{p}$ and $\hat{p}$, respectively, we obtain $\langle \check{p}, \check{A}\check{x} \rangle = \check{\lambda}$ and $\langle \hat{p}, \hat{A}\hat{x} \rangle = \hat{\lambda}$. It is obvious that the left-hand sides of these equalities represent the cost, expressed in equilibrium prices, of producing the vectors of outputs $\check{x}$ and $\hat{x}$, respectively. □

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Network program system for design optimization

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The studies have shown that the electronic circuits parameters are subject to random variations. Due to uncertainties in manufacturing process, the actual set of analog electronic circuits produced will be different that the nominal design. Even when the manufacturing process is tightly controlled so that there is little variation among the circuits produced, the environment in which the circuit will operate may not be well controlled. Similarly, aging of deterioration will cause the circuit to change in time. Thus various influences, manufacturing, environmental and aging, will cause random variations among circuits which are nominally the same.

The paper discusses the problem of choosing parameter nominals of analog electronic circuits for which the system survival probability or the performance assurance probability for the predetermined time period is maximized. Special attention is paid to parallel methods and algorithms that reduce the computation cost of stochastic optimization problems.

On the basis of the proposed parallel methods and algorithms for region of acceptability location, modeling, stochastic criterion calculation and discrete optimization [1, 2] a computer-aided reliability-oriented distributed design (CARD) system has been developed. The CARD system builds mathematical models and calculates ratings (nominal) of component parameters so that achieve the highest precision, acceptability (manufacturing yield) or reliability of analog electronic circuits under design.

The CARD system includes:

- the simulation module (it facilitates the use of a variety of simulation programs for electronic circuits design);
- the module for deterministic and statistical analysis;
- the module for objective function (reliability and/or manufacturing yield) calculation;

- the optimization module.

The system is organized from group of computers incorporated in a network. Such system allows using all advantages of client-server technology. The following tasks are executing by server:

- generation of initial circuit according to the certain requirements, or granting of convenient manual input;
- gathering information about clients, IP address and information of clients performance;
- splitting the circuit with special algorithms by pieces for modeling realization;
- transfer the data to client stations and start designing;
- reception the final optimized parts of circuit and associates it to the uniform circuit.

The following tasks are executing by clients:

- reception information from the server and sending signal about readiness for begin designing;
- start designing on basis of the chosen algorithm;
- transfer results to the server.

CARD system uses a widely distributed PSpise circuit simulation program that allows simulating a large class of analogous devices in direct current, frequency and time domains. CARD also consists of features for nominal design, design centering, tolerances assignment, etc.

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Reduction of a problem of maximizing the volume of three-dimensional bodies based on sub metrical converting to the variation calculus problem

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There is a hypothesis: flatten any convex polyhedron can always be folded non-convex polyhedron with more volume. To approach this problem, we reduce it to the problem of the calculus of variations. And it takes the form:

$$J[r,h] = \int_0^{2\pi} \int_0^1 r^2(t\prime,\alpha\prime)\dot{h}_t(t,\alpha)\ dt\prime\, d\alpha\prime \to max$$

$$\dot{r}_t^2 + \dot{h}_t^2 \le A(t,\alpha)$$

$$\dot{r}_\alpha^2 + \dot{h}_\alpha^2 + r^2 \le C(t,\alpha)$$

$$1.\ (\dot{r}_y^2 + \dot{h}_t^2 - A(t,\alpha))(\dot{r}_\alpha^2 + \dot{h}_\alpha^2 + r^2 - C(t,\alpha))) = 0,$$

$$\dot{h}_t\dot{h}_\alpha + \dot{r}_t\dot{r}_\alpha = B(t,\alpha)$$

$$2.\ (\dot{r}_y^2 + \dot{h}_t^2 - A(t,\alpha))(\dot{r}_\alpha^2 + \dot{h}_\alpha^2 + r^2 - C(t,\alpha))) = 0,$$

$$(\dot{h}_t\dot{h}_\alpha + \dot{r}_t\dot{r}_\alpha - B(t,\alpha))^2 - (\dot{r}_y^2 + \dot{h}_t^2 - A(t,\alpha))(\dot{r}_\alpha^2 + \dot{h}_\alpha^2 + r^2 - C(t,\alpha)) \le 0$$

Here h – eight along the axis of the figure, r – the distance from the geometric axis of the body to the point of its surface, α – angle of rotation around the geometric axis of the body, $A(.,.), B(.,.), C(.,.)$ – functions defining a geometric shape.

In this way it is possible to get the maximum volume of rotation figures. The solutions are obtained for the following geometric shapes: cylinder, cone, truncated cone.

The signal recovery by multiple integrals and desktop-grid computing

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One of the major problems in the control theory is the task of rebuilding the signal as a function. The most common situation where this function is either a segment of a power series , or the sum of sinusoids . Signal recovery is produced in the vicinity of the measuring point , and this point can be both continuous and discrete parameter measured process. Application of Taylor's formula and the close approaches applied control theory is complicated by the need to use derivatives. The fact is that real devices can measure not derivatives but integrals. Integrators is used for this purpose.

In this report we propose a formula which, as Taylor's formula, is a segment of a power series and allows to approximate a given function with the required degree of accuracy in the neighborhood of a given point, but its coefficients are calculated in terms of multiple integrals. A connection of this formula with the Taylor formula is studied. Also proposed a formula that allows to recover the signal as a sum of trigonometric functions.

We note some advantages that proposed in the approach to the restoration of the signal. One of the problems of classical control theory is the problem of constructing differentiators. The main complicating circumstances arising in the solution of this problem are incorrect computational differentiation, complexity of the physical implementation of this operation. In the proposed approach, it all comes down to the correct and physically realizable integration operation. Derivatives are calculated if necessary by constructing a segment of a power series with any degree of accuracy. Moreover, the proposed approach can effectively filter out high frequency noise.

The report provides an experience of desktop grid computing experiments for recovery signals. The prospects for the use of this environment for large-scale computing is discussed.

A Polynomial-Time Algorithm for the Vector Subset Selection Problem

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We consider the following problem (from now on all norms are supposed to be Euclidean):

Vector Subset Selection Problem

Instance: A set $\mathcal{X} = \{x_1, \ldots, x_N\}$ of points in $\mathbb{R}^q$ and a positive integer $L < N$;

Goal: Find a subset $\mathcal{C} \subseteq \mathcal{X}$ of cardinality L minimizing $\sum_{y \in \mathcal{Y}} \|y - \overline{y}(\mathcal{C})\|$ where $\overline{y}(\mathcal{C}) = \frac{1}{|\mathcal{C}|} \sum_{y \in \mathcal{C}} y$ is a centroid of $\mathcal{C}$.

In [2], Kel'manov and Romanenko developed a 2-approximation algorithm with running time $O(qN^2)$ for this problem.

We prove that *Vector Subset Selection Problem* can be solved exactly in $O((d+1)^2 N^{2(d+1)})$ time, which implies polynomial-time solvability of the problem when the dimension q is fixed.

Our proof is based on a polynomial-time reduction to the following problem investigated in [1]:

Instance: A set $\mathcal{X} = \{x_1, \ldots, x_N\}$ of points in $\mathbb{R}^q$ and a positive integer $L < N$;

Goal: Find a subset $\mathcal{D} \subseteq \mathcal{X}$ of cardinality L maximizing $\sum_{y \in \mathcal{D}} \|y\|$.

The work was supported by the the Russian Foundation for Basic Research (project no. 12-01-00184).

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NP-hardness of the Euclidean MAX-CUT problem

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We consider the problem in which given a complete edge-weighted undirected graph whose vertices are points of the q-dimensional space, it is required to find a cut of maximum total weight. Two special cases are analyzed where the edge weights are equal 1) to the Euclidean distances between points representing the vertices (*Euclidean Max-Cut* problem), 2) to the squares of these distances (*Quadratic Euclidean Max-Cut* problem).

Euclidean Max-Cut. *Given a set* $\mathcal{X} = \{x_1, \ldots, x_N\}$ *of points in* $\mathbb{R}^q$. *Find: a partition of the set* $\mathcal{X}$ *into two subsets* $\mathcal{Y}$ *and* $\mathcal{Z}$ *such that* $\sum_{y\in\mathcal{Y}} \sum_{z\in\mathcal{Z}} \|y - z\| \to \max$.

Quadratic Euclidean Max-Cut. *Given a set* $\mathcal{X} = \{x_1, \ldots, x_N\}$ *of points in* $\mathbb{R}^q$. *Find: a partition of the set* $\mathcal{X}$ *into two subsets* $\mathcal{Y}$ *and* $\mathcal{Z}$ *such that* $\sum_{y\in\mathcal{Y}} \sum_{z\in\mathcal{Z}} \|y - z\|^2 \to \max$.

We prove [1] that both problems 1) are strongly NP-hard; 2) do not admit fully polynomial time approximation schemes (FPTAS) unless P=NP. The proofs are similar and based on polynomial-time reductions from the NP-hard *Minimum Bisection* problem on cubic graphs.

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The method of solution to the large systems of differential equations of the block structure with unseparated boundary conditions and its applications

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We consider the system consisting of L independent subsystems of linear differential equations

$$\dot{y}^i(x) = A^i(x)y^i(x) + B^i(x), \quad x \in [0, l_i]\,, \tag{1}$$

$$y^i(\cdot) \in R^{n_i}, i = 1, ..., L.$$

Here $A^i(x)$, $B^i(x)-$ are known continuous square matrices' and vector functions dimension $n_i$accordingly, where $A^i(x) \neq const$, $x \in (0, l_i)$; unknown vector functions $y^i(x)$dimension n_i are continuous differentiable at $x \in [0, l_i]$; $l_i > 0$- are given; $i = 1, ..., L$. The solutions $y^i(x)$, $i = 1, ..., L$, to the subsystems in (1), are connected with initial and boundary conditions, which we will write in the following general form:

$$Gy(0) + Qy(l) = R, \tag{2}$$

where $G = ((g_{ij}))$, $Q = ((q_{ij}))-$are given square matrices dimension $n \times n$, $n = \sum\limits_{i=1}^{L} n_i$, and the rank of extended matrices' (G, Q) is: $rang(G, Q) = n$; $R = (r^1, ..., r^n)^T-$ is given vector dimension n. The most of elements of the matrices $G$and Q are zero in practice, and nonzero elements match to the connection between initial and final states of corresponding distinct nodes of the complex object. The problem (1), (2) is the two-point boundary value problem and is characterized by the following specific features: 1) the subsystems of differential equations of the system (1) are mutually independent, 2) the solutions $y^i(x), i = 1, ..., L$,of the subsystems are connected by unseparated boundary conditions, 3) the great number of subsystems, and consequently in general by the large order of the system

(1). The problem of calculation of unsteady fluid flows in the pipeline networks of complex structure are brought to the considered problem. The mathematical models of such processes are described by the systems of partial differential equations. These systems consist of subsystems of hyperbolic type equations, which describe the process of fluid flow in each distinct segment. The condition of thread continuous and material balance is satisfied at junctions, determined by the conditions (2). The problem of calculation of the regimes of fluid flow in the pipeline network is reduced to the problem (1),(2) by the application of straight method (analogues to the application of decomposition) [1,2].

The results of numerical experiments are given, which are obtained by the solution to the model problem, which base is the problem of calculation of unsteady fluid flow for the segment of the pipeline network of complex structure.

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On an inverse coefficient problem for parabolic equation

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We consider the following problem to determine an unknown coefficient $C_0(x)$ of linear parabolic equation:

$$\frac{\partial v(x,t)}{\partial t} = a_0(x)\frac{\partial^2 v(x,t)}{\partial x^2} + a_1(x)\frac{\partial v(x,t)}{\partial x} + a_2(x)v(x,t) + f(x,t) + \\ + B_0(x,t)C_0(x),\ (x,t) \in \Omega = \{(x,t) : 0 < x < l,\ 0 < t \le T\}, \qquad (1)$$

under the following conditions:

$$k_1 v(x,0) + \int_0^T e^{k\tau} v(x,\tau)\,d\tau = \varphi_0(x),\ v(x,T) = \varphi_T(x),\ x \in [0,\, l], \quad (2)$$

$$v(0,t) = \psi_0(t), \quad v(l,t) = \psi_1(t), \quad t \in [0,\, T]. \qquad (3)$$

Here the functions $a_0(x) > 0, a_1(x), a_2(x), f(x,t), B_0(x,t), \varphi_0(x), \varphi_T(x), \psi_0(t), \psi_1(t)$ and the constants k, k_1 are given, the functions $\varphi_0(x), \psi_0(t), \psi_1(t)$ satisfy the conformation conditions:

$$k_1\psi_0(0) + \int_0^T e^{k\tau}\psi_0(\tau)\,d\tau = \varphi_0(0), \quad k_1\psi_1(0) + \int_0^T e^{k\tau}\psi_1(\tau)\,d\tau = \varphi_0(l),$$

and all other necessary conditions of existence and uniqueness of the solution to the inverse problem (1)-(3), which consists of determining of an unknown continuous function $C_0(x)$ and corresponding solution to the problem: $v(x,t)$.

We'll obtain the equality:

$$\frac{\partial u(x,t)}{\partial t} = \tilde{a}_0(x)\frac{\partial^2 u(x,t)}{\partial x^2} + \tilde{a}_1(x)\frac{\partial u(x,t)}{\partial x} +$$
$$+\tilde{a}_2(x)u(x,t) + \tilde{f}(x,t) + \tilde{B}_0(x,t)C_0(x) + C_1(x), \qquad (4)$$

after introducing an auxiliary function $u(x,t) = k_1 v(x,0) + \int_0^t e^{k\tau} v(x,\tau)\, d\tau$ and some conversions. Here $\tilde{a}_0(x)$, $\tilde{a}_1(x)$, $\tilde{a}_2(x)$, $\tilde{f}(x,t)$, $\tilde{B}_0(x,t)$ are definite functions. So, there are two unknown functions $C_0(x)$, $C_1(x)$ in (4), and we use the conditions

$$u(x,0) = k_1 \frac{\partial u(x,0)}{\partial t}, \quad u(x,T) = \varphi_0(x),$$
$$\frac{\partial u(x,T)}{\partial t} = e^{kT}\varphi_T(x) = \tilde{\varphi}_T(x), \quad x \in [0,l], \qquad (5)$$

$$u(0,t) = k_1\psi_0(0) + \int_0^t e^{k\tau}\psi_0(\tau)\, d\tau = \tilde{\psi}_0(t),$$

$$u(l,t) = k_1\psi_1(0) + \int_0^t e^{k\tau}\psi_1(\tau)\, d\tau = \tilde{\psi}_1(t), \quad t \in [0,\, T], \qquad (6)$$

to define them. In the work we suggest the approach for the numerical solution to the problem (4)-(6), based on using of straight method. The problem is reduced to the system of ordinary differential equations with unknown parameters. We suggest a method to determine the unknown parameters, based on the sweep method [1].

We will give all the necessary calculation schemes, formulas and the results of numerical experiments on model problems, at the report.

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Application of multi-agent genetic algorithm for multi-objective optimization in large-scale systems

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There is a class of control systems having super large dimensions space of decisions. The examples of such systems are described in the works [1]-[2]. In particular, in the work [1] is represented the simulation system allowing to maximize the shareholder value of a vertical-integrated oil company. Such system consists of some sub-systems (oil production, oil transport, oil refining, etc.) and it includes more than 2,000 investment projects, which are decision variables in appropriate optimization problem. In the work [2] is represented the simulation system allowing to solve a multi-objective optimization problem of a large internet shop. For the solving such large-scale multi-objective optimization problem and forming Pareto-front, the multi-agent genetic algorithm (MAGAMO) was developed [3]. In particular, in the work [2] was formulated the following problem which was solved with the help of MAGAMO.

Problem 1. *It is necessary to define the optimal values of operative and strategic control parameters* $\{q, a_j, m1_i, m2_j, m3_k, p_j(t), d_i(t), m(t)\}$ *, which give maximum values of the profit and client's base under minimum value of the inventory turnover:*

$$\begin{cases} \max_{\{q,a_j,m1_i,m2_j,m3_k,p_j(t),d_i(t),m(t)\}} EBIDTA \\ \max_{\{q,a_j,m1_i,m2_j,m3_k,p_j(t),d_i(t),m(t)\}} CB(t) \\ \min_{\{q,a_j,m1_i,m2_j,m3_k,p_j(t),d_i(t),m(t)\}} \overline{L} \end{cases}$$

under different constraints at the each time t:
- marginality level constraint for each j-product's category;
- market share constraint for each j-product's category;
- market share constraint for each i-region;
- inventory turnover constraint.

These constraints and appropriate characteristics are described in the work [2] in more detail. Control parameters are represented by different characteristics such as the level of a quality of ordering processing, availability of products on the inventory, marketing activity by i-region, j- product's category and other factors. In the considered model are included 5 product's categories, 6 regions, 3 client segments, 52 weeks, 3 objective functions together with all constraints. Therefore, the scale of decision variables will reach approximately 380 000. It allows considering this system like large-scale multi-objective optimization problem. Such problems cannot be solved with the help of traditional methods including classic genetic algorithms even such as SPEA (Strength Pareto Evolutionary Algorithm). Therefore, the MAGAMO was applied to solve the **Problem 1**. The algorithm based on the dynamical interaction of synchronized agents which are interdepended genetic algorithms (GAs) having own separate evolutions of their populations. This approach has some similarities with well known "island model" of GA. In both methods is used a migration of individuals from agents ("islands") to the main process ("continent"). In contrast, the intelligent agents in MAGAMO are able to decompose the dimensions space to form evolutions of subpopulations between other agents by its personal rules. In the same time, the main (central) process is responsible only for the coordination of agents and their selection according Pareto rules (without evolution). Intelligent agents seek local suboptimal solutions for a global optimization, which will be completed in the result of the interaction of all agents. In the result of this, the amount of needed recalculating the fitness-functions was significantly reduced.

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Difficulties of numerical solution of separable and quasi-separable optimization problems with 10^9 variables

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Optimization problems with large and very large dimensions ("Huge Scale optimization problems") occur naturally in a wide range of scientific fields - big data analysis, analysis of genomic chains, analysis of telecommunication networks, machine learning, images recognition, optimization of atomic and molecular clusters and many others. Yu. E. Nesterov recently proposed [1] the following classification of optimization problems by number of optimized variables:

- "Small" problems – up to 100 variables
- "Medium" problems – from 10^3 to 10^4 variables
- "Large" problems – from 10^5 to 10^7 variables
- "Huge" problems – more than 10^8 variables

Steady progress of modern computer technologies, especially in parallel and hybrid architectures, and its availability expansion to a wide user gives optimism in investigation of solving optimization problems with discussed dimensions. Vigorous research of discussed problem conducted in a number of scientific organizations, both in Russia and abroad. In Belgium, serious progress on huge problems achieved by a team of Yu. E. Nesterov [2], in U.S. successfully working group of A. S. Nemirovsky [3] and several others, in U.K. – P. Richtarik group, in Russia - group of A. V. Gasnikov [4] in MIPT (PreMoLab). The number of scientific publications on this topic is growing rapidly.

This report discusses the proposed algorithms for solving some classes of multidimensional optimization problems. The authors consider the optimization of separable and quasi-separable problems: problems with

three-diagonal Hessian, problems with a dense column (row) in the Hessian, problems with random arrangement of elements in the row (column) of the Hessian, and some others. For this problems investigated the properties and behavior of some local optimization methods modifications: B.T. Polayk's method [5], Yu.E. Nesterov's methods [1], Cauchy's method, conjugate gradient methods [6], quasi-Newton L-BFGS method.

All described methods are implemented by authors with C++ language with using OpenMP and MPI [7] technologies for parallel computing. The results of numerical experiments for considered optimization problems with dimensions up to 10^9 variables, performed on local workstations and distributed high-performance cluster systems, are presented. Some computational technologies, aimed to performance and efficiency increasing of implemented optimization methods are suggested.

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Optimal parametric synthesis special features

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Reliability ensuring of technical devices and systems taking into account technological and maintenance parameters deviations from the nominal values is an actual research area. In this paper discusses new features of optimal parametric synthesis for gradual failures reliability ensuring that allow to choose an appropriate strategy when designing analog electronic technical devices and systems with deterministic structure and randomly changing parameters value during operation. A strategy means a set of methods and algorithms for solving specific optimal parametric synthesis problems corresponding to the available prior information about the parameters degradation, terms of reference and development costs estimated time.

Common to all strategies is to perform at the first stage the acceptability region preliminary analysis procedure. Its application allows to find acceptability region characteristic points that can be search optimization starting points. Preliminary procedure allows to test the hypothesis of region convexity and simply connected one and if it is not rejected to build the acceptability region inner approximation by contact points [1].

In the absence of any information about the law of parameters degradation is proposed to replace the original stochastic criterion by deterministic one. Most often used for this purpose a minimum performance capacity criterion that based on geometrical methods of parametric reliability evaluation and analysis [2]. In some cases, the original problem can be reduced to a mathematical programming problem.

If it's known laws and characteristics of the degradation parameters processes, which is determined by technological factors, the optimal parametric synthesis problem is called yield optimization problem. In this

case to ensure reliability of gradual failures can be used the combined criterion: select the most remote from the borders point from variety ones with maximum yield value.

With a sufficient amount of priori information about parameters values stochastic variations the problem at hand can be solved using the statistical modeling method and critical sections one [3].

Another possible strategy for solving the optimal parametric synthesis problem based on acceptability region construction. As one of the possible variants of the acceptability region discrete analogue construction is proposed to snap the search area to a uniform grid by means parameters values coordinates axes quantization method and array of characteristic function values generation that reduced to grid elements exhaustive search with model output parameters and performance conditions verification calculation [4]. In the parameters synthesis problem nominal values search area is a discrete itself. This is due to the fact that the parameters nominal values of the most common electro-radio components (resistors, capacitors, induced coils, operational amplifiers, and others) are regulated by the technical conditions and standards. Discrete optimization on the nominal values set is the most time-consuming of all possible solving problem strategies. But it allows to get the possible solutions full set or to stop the exhaustive process when reaching the first record.

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On optimization problem in creep

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In the design of ship hulls and aircraft are widely used panels and profiles of hardly-deformed aluminum alloys at normal temperatures. Traditional methods of formation of structural elements of such alloys often leads to the appearance of the plastic breaks, cracks and other damage. In this regard, effective way of the formation under high temperature and low deformation under creep conditions is necessary to obtained. The use of such process ensures the production of parts with high accuracy, which reduces the complexity of assembly and welding, and also reduces the formation of effort to improve the residual service life and quality of construction. In this case there is a necessity of calculation of such processes in the model with finite irreversible deformations and complicated rheological properties of materials.

The present is carried out within the model of finite elastoplastic deformations [1]. The constitutive kinematic equations are writen in the Cartesian system (Eulerian coordinates). The specified values of residual displacements are obtained in elastocreep material within a specified period of time to the final point of unloading We formulate the optimal deformation problem in the creep: we are necessary to define a way of deformation of to obtain .

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A boundary-value optimal control problem with terminal conditions in the form of extreme mappings

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The problem of computing a fixed point of extreme mapping is the most common mathematical structure which includes almost all of the known problem formulations. Namely, these are problems of convex programming, saddle and game programming problems, equilibrium programming problems, variational inequalities, and dynamic game problems.

In general, the problem of computing a fixed point of extreme mapping may be formulated as

$$v^* \in \operatorname{Argmin}\{\Phi(v^*, w) + \varphi(w), \;\; w \in W \subset R^n\}, \tag{1}$$

where $W = \{w \in R^n \mid Aw \leq a\}$ is convex, bounded polyhedron, the function $\Phi(v^*, w) + \varphi(w)$ is convex in $w \in W$, for any fixed value of the variable $v \in W$. The function $\Phi(v, w)$ is antisymmetric with respect to its variables, i.e. satisfies the condition $\Phi(v, w) = -\Phi(w, v)$.

Any positive semi-definite function, i.e. subject to the condition

$$\Phi(w, w) - \Phi(w, v) - \Phi(v, w) + \Phi(v, v) \geq 0 \;\; \forall v, w \in W \times W,$$

can be reduced to the form presented in (1). In the regular case the extreme mapping (1) always has a fixed point.

Problems of type (1) often play the role of mathematical models to describe the various objects in the economy, technology, politics, and scientific research. However, over time the objects are changed under the influence of various factors in continuously changing environment. Therefore it is interesting to consider these models in a dynamic form.

We consider the boundary value optimal control problem, where the boundary conditions are the fixed points of extreme mappings (1):

$$v_0^* \in \operatorname{Argmin}\{\Phi_0(v_0^*, w_0) + \varphi_0(w_0), \; w_0 \in W \subset R^n\}, \tag{2}$$

$$v_1^* \in \operatorname{Argmin}\{\Phi_1(v_1^*, w_1) + \varphi_1(w_1),\ w_1 \in W \subset R^n\}, \tag{3}$$

$$\frac{d}{dt}v(t) = D(t)v(t) + B(t)u(t),\ v(t_0) = v_0^*, v(t_1) = v_1^*,\ u(t) \in U. \tag{4}$$

Here, the fixed points of extreme inclusions (2), (3) are the initial and terminal conditions of the dynamical system (4). With the choice of control $u(\cdot) \in U \subset R^s$, the trajectory $x(t)$, as a solution to the differential equation (4), takes an object from state (2) to the state (3).

In the paper the saddle method is formulated, and under certain conditions its convergence to the solution of the problem is proved. In this case, the convergence in controls is weak, the convergence in direct and dual trajectories is strong, and the convergence in terminal variables also is strong.

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About functions with the Lipschitz continuous gradient

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Let H be a real Hilbert space.

We shall consider the next problems in a real Hilbert space $\mathcal{H}$:

$$\max_{x \in A} f(x), \tag{1}$$

where $f : \mathcal{H} \to \mathbb{R}$ is a continuous function, $A \subset \mathcal{H}$ is a closed convex bounded subset.

We shall say that a function $f : \mathcal{H} \supset U \to \mathbb{R}$ has the *Lipschitz continuous gradient with constant* $C > 0$ *on the set* U, if for any two points $x_0, x_1 \in U$ we have

$$\|f'(x_0) - f'(x_1)\| \leq C \cdot \|x_0 - x_1\|.$$

For a function $f : \mathcal{H} \to \mathbb{R}$ and a number α define lower level (Lebesgue's) set

$$\mathcal{L}_f(\alpha) = \{x \in \mathcal{H} \mid f(x) \leq \alpha\}$$

and upper level (Lebesgue's) set

$$\mathcal{U}_f(\alpha) = \{x \in \mathcal{H} \mid f(x) \geq \alpha\}.$$

We shall say that a closed convex subset $A \subset H$ is R-strongly convex ($R > 0$) [1] if the set A can be represented as an intersection of closed balls of radius R.

Theorem 1. *Suppose that a function $f : \mathcal{H} \to \mathbb{R}$ has the Lipschitz continuous gradient with constant $C > 0$ on the set $\mathcal{L}_f(\alpha)$. Let*

$$M = \inf\{\ \|f'(x)\| \mid x \in \partial\mathcal{L}_f(\alpha)\}.$$

Then the set $\mathcal{U}_f(\alpha)$ is proximally smooth [2, 3] with constant $R = \frac{M}{C}$.

Let $m = \inf\{\ \|f'(x)\| \mid x \in \partial A\}$ and

(i) The set A is strongly convex of radius $r > 0$,

(ii) The function $f : \mathcal{H} \to \mathbb{R}$ has the Lipschitz continuous gradient on the convex set A with constant $C > 0$,

(iii) $r < \frac{m}{C}$.

Theorem 2. *Suppose that conditions (i)-(iii) take place in the problem (1). Then the iteration process* $x_0 \in \partial A$,

$$x_{k+1} = \arg\max_{x \in A}(f'(x_k), x), \qquad k = 0, 1, 2, 3, \ldots$$

converges to the (unique) solution z_0 *of the problem with the rate of geometric progression with common ratio* $q = \frac{rC}{m} < 1$.

Theorems 1 and 2 will be published in Journal of Mathematical Sciences (New York).

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A nonnegative solution of a linear equations system

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A method for finding a nonnegative solution of linear equations system based the proposed in [1] the algorithm of selection the extreme vectors of a pointed convex polyhedral cone is proposed. The computational complexity of the proposed method polynomial depends on the dimension of the problem.

Consider the problem of finding $x^* \in \mathbb{R}_n$, satisfying the system

$$Ax = b, \quad x \geq 0, \tag{1}$$

where A is the matrix of dimension $m \times n \quad (m < n)$ of rank m, vector $b \in \mathbb{R}_m$. It is natural to assume that $b \neq 0$.

It is obvious that the system (1) is solvable, if vector b belongs to the cone $K = \{u \in \mathbb{R}_m \mid u = Ax, \quad x \geq 0\}$. Assume that the cone K is pointed. Consider the cone V, stretched on the projection of the vector-columns of the matrix A on the hyperplane $L(b) = \{x \in \mathbb{R}_m \mid \langle b, x\rangle = 0\}$. If the cone V is pointed, that as follows from [1], vector $b \notin K$, i.e. the system (1) has no solutions. If the cone V is not pointed, that any solvable system (1) or system $Ax = -b, \quad x \geq 0$. In the latter case, the following algorithm is going to solve this system, where would follow that the system (1) insoluble.

Now, let the cone K is pointed. Using the proposed in [1] algorithm, you select all extreme direction vectors of the cone K and remove all vectors that are not extreme. In other words, we can assume that all direction vectors of the cone K are extreme. Suppose that $b \in K$. Then this vector together with any set of direction vectors of the cone K forms a pointed cone, as contained in pointed cone. Let us take an arbitrary index $s \in J = \{1, \ldots, n\}$ and consider the cone

$$K_s = \{u \in \mathbb{R}_m \mid u = \bar{A}x, \quad x \geq 0, \quad x \in \mathbb{R}_{n-1}\},$$

where $\bar{A} = [A_1, \ldots, A_{s-1}, A_{s+1}, \ldots, A_n]$. If vector b is extreme in cone

$$K_s^+ = \{u \in \mathbb{R}_m \mid u = \bar{A}x + by, \quad x \geq 0, \quad x \in \mathbb{R}_{n-1} \quad y \geq 0, y \in \mathbb{R}_1\},$$

(this can be determined using the algorithm from [1]), then $b \notin K_s$. If $b \in K_s$ then it is obvious that in decision the problem (1) the variable x_s^* can be put equal to zero, i.e. a column vector A_s can be excluded from the matrix A. Obviously also, that if you remove any vector A_s cone K_s remain as pointed and all direction vectors of this cone remains the extreme.

This gives the opportunity to formulate an algorithm for solving the problem (1). In fact, we consider the matrix $\bar{A}$, consistently excluding from the original matrix A of vector-columns starting with the first if the vector b belongs to the cone, stretched the remaining vector-columns. In this case the original problem (1) is reduced to the similar problem where the dimension of the vector x one less. Reducing thus the initial problem into the problem of smaller dimension, get regular square matrix a $\bar{A}$, for which $b = \bar{A}x, \quad x \geq 0$. Thus, we arrive at the following statement.

Theorem 1. *Let the cone K is pointed, $b \in K$. Then the algorithm provides the solution of problem* (1) *for $N = O(n^4)$ elementary operations.*

Indeed, as of about [1], the procedure relating to regarding the facilities of the vector b of the cone K_s, requires $O(n^3)$ operations. The calculation of the rank of the matrix and the inverse of the matrix — the procedure is the same computational complexity. Because in the process of solving tasks in accordance with the algorithm the number of cycles does not exceed n, then the total the computational complexity of the algorithm is estimated value $N = O(n^4)$.

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On Euler-Lagrange equations with non-B_u-potential densities of forces

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Consider the equation

$$N(u) \equiv P_{2u,t}u_{tt} + P_{1u,t}u_t + P_{3u,t}u_t^2 + Q(t,u) = 0, \qquad (1)$$

$$u \in D(N) \subseteq U \subseteq V,\ \ t \in [t_0,t_1] \subset \mathbb{R},\ \ u_t \equiv D_t u \equiv \frac{d}{dt}u,\ \ u_{tt} \equiv \frac{d^2}{dt^2}u.$$

Here $\forall t \in [t_0,t_1], \forall u \in U_1\ P_{iu,t} : U_1 \to V_1$ $(i = \overline{1,3})$ are linear operators; $Q : [t_0,t_1] \times U_1 \to V_1$ is an arbitrary generally nonlinear operator; $D(N)$ is the domain of definition of the operator N; $U = C^2([t_0,t_1];U_1)$, $V = C([t_0,t_1];V_1)$, U_1, V_1 are real linear normed spaces, $U_1 \subseteq V_1$.

We follow notations and terminology used in [1]—[3].

Necessary and sufficient conditions for the representation of the given operator equation in the form of the Euler-Lagrange equation with non-B_u-potential density of force are obtained and the corresponding functional is constructed. In some special cases problems of minimization of the functionals are investigated.

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Hopf bifurcation in reaction diffusion systems

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This work focuses on the application of abstract bifurcation theorems to concrete dynamic system described by ODE and PDE. We formulate an abstract bifurcation task and give classical and contemporary abstract theorems about bifurcations.

Also, there is described how the abstract results are applied to prove the existance of periodic solutions in system of ODE. In this way we prove the classical Hopf bifurcation theorem, as well as some important generalizations of it.

In the second part od work we apply the abstract theorems to examine the periodic solutions of reaction-diffusion systems. This application requires complex mathematical techniques because the dynamic systems are infinite-dimensional.

Some attention we also give to applications of reaction-diffusion systems for describing the processes known in chemestry and biology.

Optimal transmission capacity for two-node market

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We examine a two-node competitive market. The goal is to maximize the total welfare and to determine the optimal transmission capacity. Consumers at the node $i = 1, 2$ are described by a non-increasing demand function $d_i(p)$, $d_i(p) \to 0$ as $p \to \infty$. There is a final set of producers A_i at the local market i. Each producer a is characterized by convex cost function $E^a(q)$. Loss coefficient k determines the share of the good, lost under transmission; C is the be transmition capacity.

Strategy of producer a is a non-decreasing supply function $r^a(p)$ that determines the output volume depending on the price p. First, the network administrator computes clearing prices $\overline{p_i}$ for isolated markets. These prices are determined by the equations $\sum_{a\in A_i} r^a(p_i) = d_i p_i, i = 1, 2$. If $1-k \leqslant \overline{p_2}/\overline{p_1} \leqslant (1-k)^{-1}$, there is no transmission from one market to the other. Otherwise let $\overline{p_2}/\overline{p_1} \leq (1-k)^{-1}$. Then the network administrator determines the volume of good v that will be transmitted from the market 1 to market 2. Under a given flow, v the nodal prices $p_1(v)$ and $p_2(v)$ meet the equations $\sum_{a\in A_1} r^a(p_1) = d_1 p_1 + v, \sum_{a\in A_2} r^a(p_2) = d_2 p_2 - (1-k)v$.

There are two possible outcomes. The first one corresponds to the conditions: $(1-k)^{-1}p_1(v) = p_2$ and $v \leqslant C$. Otherwise $v = C$ and $(1-k)^{-1}p_1(v) < p_2$.

Under assumption of perfect competition, the optimal strategy of each producer is determined by Walrasian supply function $s^a(p) = \arg\max_{q^a}(q^a p - E^a(q^a))$

Let $\widetilde{p_i}(Q), i = 1, 2$ denote the prices corresponding to Walrasian supply functions. Prices $\widetilde{p_i}(0)$ meet the equations $s_i(\widetilde{p_i}) = d_i(\widetilde{p_i}), i = 1, 2$. We assume that $(1-k)\widetilde{p_2}(0) > \widetilde{p_1}(0)$. If there is a flow from the first market to the second market, the prices satisfy the following conditions

$$s_1(\widetilde{p_1}) = d_1(\widetilde{p_1}) + v, s_2(\widetilde{p_2}) = d_2(\widetilde{p_2}) - (1-k)v \qquad (1)$$

$v \leq C, \widetilde{p_1} = (1-k)\widetilde{p_2}$ or $v = C, \widetilde{p_1} < (1-k)\widetilde{p_2}$

Let functions $\widetilde{p_1}^0(v)$ and $\widetilde{p_2}^0(v)$ be determined by (1). If $v = C$, then $\widetilde{p_i}(C) = \widetilde{p_i}^0(C)$.

Theorem 1 *There exists a value of the transmission capacity* $\widehat{C}$ *for which* $\widetilde{p_1}^0(\widehat{C}) = (1-k)\widetilde{p_2}^0(\widehat{C})$ *. If* $C \geq \widehat{C}$*, in equilibrium* $v = \widehat{C} < C, \widetilde{p_i}(C) = \widetilde{p_i}^0(\widehat{C}), i = 1,2$*. Otherwise,* $v = C, \widetilde{p_i}(C) = \widetilde{p_i}^0(C), i = 1,2$ $\widetilde{p_1}(C) < (1-k)\widetilde{p_2}(C)$

Without taking into account the construction costs, the welfare $N(C)$ includes the benefit of the network system $T(C)$, consumer surplus $S_i(C)$ and producers profit$P_i(C), i = 1,2$. The benefit of the network system is determined by $T(C) = \widetilde{p_2}(C)(1-k)C - \widetilde{p_1}(C)C$, if $C < \widehat{C}$. Otherwise it equals zero. Producers at market i get the total profit equal to

$$P_i(C) = \sum_{a \in A_i} ((\widetilde{p_i}(C)s^a(\widetilde{p_i}(C)) - E^a(s^a(\widetilde{p_i}(C)))))$$

.

Consumer surplus at market i is given by $S_i(C) = \int_{\widetilde{p_i}(C)}^{\infty} d_1(p)dp$. Let $B(C)$ denote the costs of the transmission line construction: $B(C) = 0$, if $C = 0$; $B(C) = b_f + b_v(C)$, if $C > 0$, where $b_v(C)$ is a convex and increasing function. The total welfare is $W(C) = N(C) - B(C)$.

Theorem 2 *Function* $N(C)$ *is concave and increases in* C*, if* $C \leq \widehat{C}$*. In addition,* $N^{'}(C) = (1-k)\widetilde{p_2}(C) - \widetilde{p_1}(C)$*.*

Theorem 3 *The optimal transmission capacity* $C^* = 0$*, if* $(1-k)\widetilde{p_2}(0) - \widetilde{p_1}(0) \leq b_v^{'}(0)$*. Otherwise a local maximum* C^{*L} *is determined by* $(1-k)\widetilde{p_2}(C^{*L}) - \widetilde{p_1}(C^{*L}) = b_v^{'}(C^{*L})$*. If* $W(C^{*L}) > W(0)$ *then* $C^* = C^{*L}$*. Otherwise,* $C^* = 0$*.*

Analysis of the multifactor business model of insurance company

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The reinsurance strategy, optimal dividends and potential investments are important decisions in different areas of the financial activities of the insurance company. The consequences of these decisions can be very powerful. That's why need to development complex models, methods and algorithms for decisions making becomes more and more urgent.

We consider analysis and optimization of the discrete model of the financial balance, where s is the constant initial capital, $c(t)$ - premiums and X_t - random non-negative aggregate amount of the loss for the t time. Simulation of the time-dependence of premiums can be achieved by using vector of loadings $\theta(t)$ at each time point $c(t) = (1 + \theta(t))EX_t$.

Reserves of the company at the time $t \geqslant 0$ will be equal to $R(t)$:

$$R(t+1) = R(t) + c(t+1) - X_{t+1},\ R(0) = s. \tag{1}$$

$d(t)$ - dividend payments at the beginning of $t + 1$ period for the financial balance of the company at the time t, $t \geqslant 0$, so the formula for finding the reserves at time $t + 1$ modified as follows

$$R^d(t+1) = R^d(t) - d(t) + c(t+1) - X_{t+1}. \tag{2}$$

Reinsurance is the possibility to shared responsibility, it is one of the most important instruments to influence the structure of the portfolio and limit insurance risk. Managers replace some unknown costs for fixed one - premium for the reinsurance. This is the way to reduce risk and enhance financial stability of the insurance company.

Reinsurance strategy is fully defined by the functions of risk sharing. $X_t = Y_t + Z_t$ is an insurable risk, $Y_t = g(X_t)$ describes part of the risk, that remains to the assignor, and $Z_t = h(X_t)$ - transferred to reinsurance.

We consider $\xi(t) > \theta(t)$ as the time-dependent parameters of load security used for calculation of flow of the reinsurer premiums. So $c_{re}(t) =$

$(1+\theta(t))EX_t - (1+\xi(t))EZ_t$, where $(1+\xi(t))EZ_t$ is flow of premiums paid to the reinsurer.

With the ability to choose reinsurance strategy the reserves of the company will be equal to:

$$R^d_{re}(t+1) = R^d_{re}(t) - d(t) + c_{re}(t) - g\left(X_{t+1}\right). \tag{3}$$

Additional condition on the probability of bankruptcy with the final planning horizon T_{fin}:

$$\psi^d_{re}(s) = P\left\{\tau^d_{re} \leqslant T_{fin}\right\}. \tag{4}$$

where τ^d_{re} is the time of ruin.

Using the quantile criterion, we find the parameters of the model for a given confidence level $0 < \alpha \leqslant 1$ of ruin probability : $\psi^d_{re}(s) \leqslant \alpha$.

The total discounted value of dividends paid on the planning horizon T_{fin} will be maximized subject to the restrictions on the probability of ruin:

$$T = \begin{cases} \tau^d_{re}, & \text{if } \tau^d_{re} \leqslant T_{fin}; \\ T_{fin}, & \text{if } \tau^d_{re} > T_{fin} \end{cases}, \tag{5}$$

$$u(s,\alpha) = E\left[\sum_{t=0}^{T-1} v^t d(t)\right] \longrightarrow \max_{d(t)\ ,g(t)\ ,\ t\in[0,T]\ ,\ \psi^d_{re}(s)\leqslant\alpha\ ,\ 0<\alpha\leqslant 1}. \tag{6}$$

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Improper linear programming problems

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Inconsistent (improper) model doesn't allow to obtain substantial information about investigated process or phenomenon directly. For this purpose the model needs to be adjusted and corrected. Types and methods of correction can be various. The most general form of correction is to change the coefficients of the left and right sides of the corresponding equations and inequalities. Corresponding correction is called matrix one. Systematic investigations of linear and convex programming improper problems were started on 70s of the last century by I.I. Eremin and his followers. These authors studied the classification problem, constructed and investigated dual theory. At the same time discrete approximations of solutions so-called committee constructions were introduced and studied. Note that the correction by right side vector of constraints and coefficients vector of object function were considered in the most investigations.

Firstly, the matrix correction was considered in the works of A.A. Vitolin. His studies were continued by V.A. Gorelik and his followers. The results of A.A. Vatolin were clarified by these authors. But, a lot of problems in matrix correction methods stay unsolved. The first problem concerns non-uniqueness of matrix correction problem solution.

The importance of improper mathematic objects investigation was showed by I.I. Eremin, in particular, he displayed, that evolution in mathematical models and problems classes theory tended to the weakening of requirements imposed on investigated mathematical object.

The sequence of statements of problems appears: the solution uniqueness and stability; uniqueness, instability (ill-posed problems); non-uniqueness and instability; improper problems; improper problems and poor formalizability; flexible modeling etc.

Linear programming (LP) problems were successfully investigated by

many authors, for example [1]. In [2] the approximate method for solving incorrect (LP) problems was suggested. Consider following problem:

$$\begin{array}{l}(C,X)\to\max,\quad A_1X=B_1,\quad X\geq 0,\quad C>0,\\ X\in R^n,\ C\in R^n,\ A_1\in R^{m\times n},\ B_1\in R^m,\ A>0,\ B_1\geq 0.\end{array}\tag{1}$$

Describe (1) in the form depending on scalar parameter t

$$\begin{array}{l}t\to\max,\quad A_2X=B_2,\quad X\geq 0,\\ A_2^T=[C\ \ A_1^T],\quad B_2^T=[t\ \ B_1^T].\end{array}\tag{2}$$

The problem (2) can not be solved. In this case the generalized LP problem is considered

$$t_1\to\max,\quad A_2^TA_2Y=A_2^TB_2,\quad Y\geq 0.\tag{3}$$

The system (3) is always consistent. Instead of the problem (3) we consider the following problem:

$$\begin{array}{l}t\to\max,\quad A_\delta Z_\delta=B,\quad Z_\delta\geq 0,\\ A_\delta=A+\delta E,\quad A=A_2^TA_2,\quad B=A_2^TB_2.\end{array}\tag{4}$$

It is known that $\lim\limits_{\delta\to 0}Z_\delta=Z_0\in Y$, where Y is the set of solutions of the problem (3). The solution of the linear system (4) is a linear function of the parameter t

$$Z_\delta=C_1+tC_2,\quad C_1,C_2\in R^n.\tag{5}$$

The condition $Z_\delta\geq 0$ ensures the validity of the inequality

$$t_1\leq t\leq t_2.\tag{6}$$

If the solution is not bounded above, then $t_2=\infty$. Thus, the appreciation of solutions is reduced to solving incorrect linear system. The system (5) was solved by the principal components method of factor analysis.

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Application of principal component method for solving incorrect linear system

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In this paper the methods of factor analysis are applied for solution of incorrect linear algebraic equations system (SLAE). It is known, the principal component method [1] in factor analysis leads to an unique solution. The initial system can be inconsistent, ill-conditioned or degenerate. Then Tikhonov regularization method is applied to this system. The problem of principal components requires determining eigenvalues of transformation matrix of linear system. As a result, SLAE solution is written as a linear combination of selected factors. The proposed approach allows to choose arbitrary regularization parameter tending to zero. Accuracy of solution depends on the accuracy of computing eigenvalues of SLAE matrix.

The task of the principal components analysis has four basic versions:

1. to approximate a data by linear manifolds of lesser dimensions;

2. to find subspaces of smaller dimension such that scatter of data (i.e the standard deviation from the mean) in the orthogonal projection on these subspaces is maximal;

3. to find subspaces of smaller dimension such that mean square distance between the points in the orthogonal projection on these subspaces is maximal;

4. for given multidimensional random variable to construct such an orthogonal coordinate transformation so that the correlations between coordinates will be equal to zero.

Consistency of the system

The system $Ax = B$ can be made consistent by disturbance of its right side:

$$Ax = B + \delta B_1, \quad \delta > 0,\ B_1 \in R^m, \tag{1}$$

where δ is a small parameter.

In addition, we can consider a normal system for system $Ax = B$:

$$A^T Ax = A^T B. \tag{2}$$

It is known that the system (2) is always consistent. An approach to determining normal solutions is proposed in [2].

Application of Tikhonov regularization method to system (2) leads to solution of the system:

$$(A_0 + \delta I)z = B_0,\ \delta > 0,\ A_0 = A^T A,\ B_0 = A^T B,\ A_1 = A_0 + \delta I, \quad (3)$$

where δ is a regularization parameter, I is the identity matrix.

Then the solution z can be written in the form:

$$z = \sum_{i=1}^{m} \beta_i F_i, \quad m < n,$$

where F_i are the factors obtained by applying the principal components method of factor analysis. These factors satisfy following relations: $AF_i = \lambda_i F_i$, $(F_i, F_j) = \delta_j^i$, $i, j = 1, \ldots, m$.

Denote $\Lambda = \sum_{i=1}^{m} \lambda_i$, $\Lambda > 0$. Then we have

$$Az = \sum_{i=1}^{m} \beta_i \lambda_i F_i, \quad \Lambda Az = B \sum_{i=1}^{m} \lambda_i.$$

Because of the uniqueness of the solution by the principle components method

$$\sum_{i=1}^{m} \beta_i F_i = \frac{m}{\Lambda} B.$$

Multiplying the both sides of the above equation by F_i, $i = 1, \ldots, m$ we obtain

$$\beta_i = \frac{m}{\Lambda}(B, F_i), \quad i = 1, \ldots, m.$$

Conclusions

The generalized method for solving incorrect linear algebraic equations system is proposed. The obtained result is new.

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Interactive techniques for numerical optimization with "expensive" cost functions

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The talk is devoted to an interactive manning of numerical optimization for problems with "expensive" cost functions [1], based upon the work-in-progress solver OPTCON-T. We propose the following conceptual technique for searching hard computable solutions:

1. By means of a certain stochastic algorithm we fill the database of initial reference values of the cost function.

2. Based on a modified Shepard function [2], we design an initial "rough" approximation of the original problem.

3. By applying global search algorithms (Grid-Based Search, Luus–Jaakola Heuristics, "Parabolic Algorithms", "Spherical search", and "Covering Methods"), we refine the initial problem approximation and localize solutions within attraction basins of extremal points of the approximated cost function.

4. Finally, we employ local search (Gradient-Based Algorithms, in particular, Conjugate Gradient Method, "Curvilinear Spherical Search", Powell and Rastrigin Methods) for reaching a desired accuracy.

Note that all the algorithms, encompassed by the OPTCON-T solver, admit effective implementation with the use of parallel processors (this enables one to increase the algorithms' efficiency).

When dealing with real-life problems, we are to assume a crucial role of the solver's Operator (User). In order to reduce time and computational expenses, our solver OPTCON-T is equipped with a graphical module. The graphical module provides the User with current results of calculation, that ensures the Operator's interactive participation in a computational process. The graphical module includes: $2D$-projections of Shepard approximations, $1D$-projections with respect to random directions, "Ridges and Concentrations Maps", $1D$- and $2D$- "Diagrams of

Testpoints Databases" etc. A priori information about the problem along with current Operator's decisions (such as control of the sequence of applied algorithms, initialization, adaptation of experts' knowledge) can be taken into account by means of a special dialog mode.

We conclude our talk with a discussion of numerical experiments. The results let us make the following observation: Automatic algorithms for global search, implemented within the OPTCON-T solver, prove themselves rather effective for optimization problems with ravine unimodal, and multiextremal functions. Unfortunately, the accuracy of such automatic calculations is not satisfactory, as a rule. On the other hand, "quasi-extremal" points, produced by the algorithms, usually appear to belong to attraction basins of extremal points of a costs function. The latter fact makes possible to increase the accuracy by further application of algorithms for local search.

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Equilibrium model for transportation system and land-use

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We propose model for joint transportation system and land-use equilibrium. In this model there are two classes of agents. First class describe transportation system users - peoples. Second class represent developers.

Suppose that we have transportation network $\Gamma(V, E)$, where V is the set of nodes and E is the set of arcs. We will work in stable dynamic model. In this model every arc e have two haracteristics: capacity $\bar{f}_e$ and free flow travel time $\bar{\tau}_e$. In equilibrium all paths for OD-pair, which are used, have the same cost. Usually OD-matrix is given. In our case, we propose that common number of people is N, but everyone of them could chose not only path for current OD-pair, but also origin and destination.

For that reason we consider extended stable dynamic model. At equilibrium there is no agent, who can gain more by switching path or by changing origin or destination.

Suppose that one have the masterplan of the city. That means that topology $\Gamma(V, E)$ of transportation system is given. Denote by s superorigin node and by t superdestination node. Superorigin and superdestination nodes are connected with original nodes by fictitious arcs. Capacity level of fictitious arcs represent number of living and working places in corresponding nodes. Cost, associated with fictitious arcs could be interpretered as rent payments, salary, parking tolls and so on.

Developers can choose development level $\bar{f}_e$ for every arc e and starting prices $\bar{\tau}_e$. Development level is restricted and become capacity constraint for corresponding arc.

After developers move regular agents behaving selfishly convegre to (agent) equilibrium.

Denote by (f^{eq}, τ^{eq}) equilibrium point for stable dynamic model ([1]).

Then outcome for developer for fictitious arc $e, e = (s, i)$ is $f_e^{eq}\bar{\tau}_e - \bar{f}_e c_e$, where c_e is cost of construction.

In the same way we can denote outcome for developers of original arcs (roads) and fictitious arcs $e, e = (i, t)$ by $-\bar{\tau}_e \cdot f_e^{eq} - c_e \cdot \bar{f}_e$.

We think that number of agents do not change.

Then the solution $(f*, \tau*, \bar{f}*, \bar{\tau}*)$ of the following optimization problem is equilibrium in stable dynamic model $(f*, \tau*) = (f^{eq}.\tau^{eq})$ and Nash equilibrium for developers $(\bar{f}*, \bar{\tau}*) = (\bar{f}^{eq}, \bar{\tau}^{eq})$.

$$\max_{\bar{f}_e, \bar{\tau}_e, e \in E_1} \min_{\bar{f}_e, \bar{\tau}_e, e \in E_2} \min_{\bar{f}_e, e \in E} \min_{f} \sum_{e \in \hat{E}} f_e \cdot \bar{\tau}_e + \sum_{e \in E} \bar{f}_e \cdot c_e + \tag{1}$$

$$+ \sum_{e=(i,t), i \in D} \bar{f}_e \cdot c_e - \sum_{e=(s,i), i \in O} \bar{f}_e \cdot c_e$$

$$\mu \leq \bar{f} \leq \gamma$$

$$0 \leq f \leq \bar{f}$$

$$\sum_{i \in O} f_{(s,i)} = N$$

$$\sum_{i \in D} f_{(i,t)} = N$$

$$\sum_{i:(i,v) \in E} f_{(i,v)} - \sum_{i:(v,i) \in E} f_{(v,i)} = 0$$

Here O, D -are the sets of origin and destination nodes. E_1 is the set of fictitious arcs, adjoint to superorigin s, E_2 is the set of fictitious arcs, adjoint to superdestination t.

Note that this is convex optimization problem with good structure. So one could use standart techniques to solve it.

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Optimal control for technical systems modelled by differential inclusions

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When studying questions of optimization of dynamic systems an actual problem is research of optimal control of the systems described by differential inclusions. Systems of this type find broad application in applied problems of control of technical systems, in particular, control of transport space systems, and also in problems of traffic control of a body in the conditions of uncertainty [1]–[5]. Questions of optimization of dynamics of controlled systems are studied in [1], [2] and in the other works. Existence and stability of differential inclusions are considered in works [6]–[8].

In the present work the problem of optimal control in which the controlled object moves in the vertical plane from the initial point in final with intermediate achievement of the set height is considered. Thus object motion on the plane breaks into two stages. At the first stage motion occurs to constant vector traction (p, q); at the second stage with constant traction $(-r, s)$. Values of positive parameters p, q, r, s get out of intervals (p_1, p_2), (q_1, q_2), (r_1, r_2), (s_1, s_2) respectively. The object is affected by the force of gravity with constant acceleration g; resistance of the environment is not considered. Object motions at the first and second stages are described by differential inclusions [5]. We consider also the modifications of models described in [3], [5].

The problem of optimal control consists in a choice of parameters p, q, r, s and values t_1, t_2 so that object motion from the initial point in final with intermediate achievement of height was carried out with the minimum fuel consumption. The criterion of quality has the form:

$$\forall\, p \in (p_1, p_2),\ \forall\, q \in (q_1, q_2),\ \forall\, r \in (r_1, r_2),\ \forall\, s \in (s_1, s_2),$$

$$\int_0^{t_1} (p + q)dt + \int_{t_1}^{t_2} (r + s)dt \to \min\,.$$

By aid of results of works [6]–[8] stability of differential inclusions is analysed. Optimal control for unambiguous parameters is defined. The algorithm is developed for a control system on object motion to the purpose in three-dimensional space providing that the purpose moves randomly with a limited speed on the plane. Algorithms of finding of optimal control the trajectories similar to the developed are offered. Series of computing experiments in Mathematica system are carried out. Modifications and generalizations of studied systems on a case of multiple-valued parameters are considered. The specified systems can be used when moving freight by means of the aircraft in conditions when achievement of a final point on a small vicinity of a surface of the earth is impossible. Further studying of systems of the described type with use of the principle of a maximum is planned.

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Optimal stabilization and estimation of linearization errors for nonlinear dynamic models of technical manipulators

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The problems of optimal stabilization and estimation of linearization error are actual problems in research of behavior of nonlinear controlled systems [1]–[9]. Some methods for solving of the problem of optimal stabilization to respect to all variables and to a part of phase variables for multiply connected nonlinear controlled dynamic systems are given in [6], [7].

The methods of solving of optimal stabilization problem for systems of ordinary differential equations are based on the fundamental results of V.V. Rumyantsev [1] about optimal stabilization of nonlinear system under the condition of minimization of a functional characterizing the quality of control.

The report is devoted to consideration of conditions of optimal stabilization and estimation of linearization error for nonlinear dynamical models of technical manipulators. Indicated models are described by nonlinear controlled multiply connected systems of ordinary differential equations. The basis of general scheme of stabilization for multiply connected systems is two-level stabilization [10]. Conditions of optimal stabilization of programmed motion for two types of nonlinear dynamical models of technical manipulators are obtained. The properties of stabilizing control and form of integrand in criteria of control quality are used taking into account that subsystems are asymptotically stable. Optimal control is synthesized at the level of the initial system.

In cases where the system of first approximation is nonlinear, an interesting problem is finding of the maximum deviation with the same initial data for solutions of the original system and its first approximation. Construction of error estimates has practical importance in problems of stability of complex technical systems. In the report we consider the construction of estimates for errors of linearization of essentially nonlinear systems. The obtained estimates are applied to the models of technical

manipulators. The main research method is the method of Lyapunov vector functions [11], [12]. The results can be used in problems of control of motion of complicated spatial mechanisms, and also in problems of stabilization of motion of multiply connected systems of different types.

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Gradient-free optimization methods with ball randomization

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In such applications as bi-level optimization and huge-scale optimization we can encounter a situation when we can not calculate subgradient of a convex function which is minimized. In such situations we can use stochastic approximation of the gradient of the form $\frac{f(x+\mu u)-f(x)}{\mu}u$, where u – is a random vector with known distribution and μ is a smoothing parameter. In [1] a normal randomization is considered, when the vector u has normal distribution. In this work we consider the case when the vector u is uniformly distributed over a unit sphere. Such randomization was considered in [2] in application to multi-armed bandits. This kind of randomization could be more convenient in some cases since the vector u is bounded in norm. We provide complexity bounds for simple random search for smooth convex function and strongly convex smooth convex function. Also we consider the case when the value of the function is calculated with independent stochastic error with zero mean and bounded by the value δ.

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On the Classification of Affine Control Systems

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Affine control systems are nonlinear systems that are linear in controls; that, is, these are the systems of the form

$$\dot{y} = f_0(y) + f(y)u, \quad y \in M \subset R^n, \quad u \in R^r. \tag{1}$$

Here, y are the phase variables; u are the controls; and M is the phase space that is a domain. We assume that f_0 is a smooth vector field; f is an n-by-r matrix in which the columns f_α, $\alpha = 1, 2, \ldots, r$, are smooth vector fields; and $\operatorname{rank} f(y) = \text{const}$. A solution (or phase trajectory) of system (1) is defined as a continuous piecewise smooth function $y(t)$ for which there exists a piecewise continuous control $u(t)$ such that $y(t)$ and $u(t)$ satisfy (1).

Along with (1), consider the system

$$\dot{x} = g_0(x) + g(x)v, \quad x \in L \subset R^n, \quad v \in R^s, \tag{2}.$$

We say that systems (1) and (2) are equivalent if there exists a diffeomorphism $\psi\colon M \to L$ of phase spaces such that ψ and ψ^{-1} take the solutions of one systems to the solutions of the other. In general, the solutions $y(t)$ and $x(t)$ correspond to different controls $u(t)$ and $v(t)$.

It is appropriate to reduce the control system (1) to an equivalent system (2) if the latter is simpler in form. For example, a complicated nonlinear system (1) can be equivalent to a linear system (2). In this case, nonlinearity is a 'causal trait', which vanishes in an equivalent system.

Vital properties of control systems like controllability, stability, and optimality of solutions are preserved in changing over to an equivalent system. Therefore, it is natural to solve any control problem for a simpler equivalent system and then 'carry over' the results thus obtained to the initial system through an diffeomorphism.

The concept of equivalence of affine systems defines an equivalence relation on the class of affine control systems. Consequently, this class is partitioned into equivalence classes consisting of equivalent systems. As usual, the problem that arises in such cases is classification, which consists in describing the equivalence classes, i.e., in describing control systems to within equivalence.

In a broader sense, this problem also includes, for example, the determination of equivalence criteria for two systems, the construction of diffeomorphisms

implementing equivalence, and the construction of the representatives (in as simple form as possible) of equivalence classes.

As usual, to solve the classification problem, one should find invariants, that is, quantities that are preserved when passing to an equivalent system. In our case, the invariants are the rank of some distributions and codistributions (Pfaffian systems) [1,2]. We use these invariants for constructing classification of some types of affine control systems.

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Evolutionary approach for selection single equilibrium state in Backman's model.

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Let $G = \{V, E\}$ be oriented graph with set of edges E, and set of nodes N. Denote w - origin-destination pair, set of all pairs W. Then P_w - set of path numbers for w and $P = \cup P_w$ set of all paths in graph. Let $X = \{x = (x_1, ..., x^{|W|}) \in R^n_+ : \sum_{k \in P_w} x_k = d_w\}$. It is necessary to introduce utilization function $u_e(x) = \sum\limits_{p \in P} \delta_{ep} x_p$, and $\delta_{ep} = 1$, if $e \in p, \delta_{ep} = 0$, else. Cost function for edge is $\tau_e(u_e(x))$, with $\tau'(\cdot) > 0$. Using our nomenclature we can introduce Beckmann model:

$$\Psi(x) = \min_{x \in X, f = \Theta x} \sum_{e \in E} \int\limits_0^{u_e(x)} \tau_e(z) dz, \tag{1}$$

where Θ - restriction matrix for balance condition, $\sigma_e(x) = \int\limits_0^{u_e(x)} \tau_e(z) dz$.

Let's look at the problem (1) in terms of evolutionary game theory. Each driver is a player, each route is a strategy, each correspondence is a population. Then $F_i(x) = -\sum\limits_{e \in E} \tau_e(u_e(x))$ is a cost function of (1).

Definition 1. *State space for finite-dimension game will be:* $X^N = \{x \in X : Nx \in Z^n\}$*, then finite-population congested game* $F^N : X^N \to R^n$ *has potential function:* $f^N(x) = -\sum\limits_{e \in E} \sum\limits_{k=1}^{Nu_e(x)} \tau_e(\frac{k}{N})$.

It is important to note that dual model to Beckmann model with entropy regularization is a model with logit-choise dynamics. In population game logit-choise model is a full support exponential protocol.

Theorem 1. **[2]** *Let* F^N *be a finite-population potential game with potential function* f^N*. Suppose that agents are clever and follow a direct exponential protocol with noise level* η*. Then the stochastic markov evolutionary process* $\{X^N_t\}$ *is reversible with stationary distribution*

$$\mu^N_x = \frac{1}{K^N} \frac{N!}{\prod\limits_{k \in P} (Nx_k)!} \exp(\eta^{-1} f^N(x)), \tag{2}$$

for $x \in X^N$*, where* K^N *is necessary for* $\sum_{x \in X^N} \mu^N_x = 1$.

Now introduce logit potential function $f^\eta : X \to R$: $f^\eta(x) = f(x) - \eta \sum_{p \in P} x_p \log x_p$.

Theorem 2. **[2]** *Let F^N be a sequence of finite-population potential games whose rescaled potential function $\{\frac{1}{N} f^N\}_{N=N_0}^{\infty}$* converges uniformly to the C^1 function $f : X \to R$. Suppose that agents are clever and employ a direct exponential protocol with noise level $\eta > 0$. Then the sequence of stationary distributions satisfies

$$\lim_{N \to \infty} \max_{x \in X^N} [\frac{\eta}{N} \log \mu_x^{N,\eta} - \Delta f^\eta(x)] = 0, \tag{3}$$

where $\Delta f^\eta(x) = f^\eta(x) - \max\limits_{y \in X} f^\eta(y)$.

Solution of task (1) does guarantee uniqueness of f^*, when $\tau'(\cdot) > 0$, but not uniqueness of x^*. There is a method of selection of unique equilibrium proposed by Bar-Gera [1], with $X^* = \{x : \Theta x = f^*, x \in X\}$:

$$\max_{x \in X^*} \sum_{p \in P} x_p \ln x_p, \tag{4}$$

It is known what: task (1),(4) and task:

$$\Phi(x) = \min_{x \in X, f = \Theta x} \sum_{e \in E} \sigma_e(x) + \eta \sum_{w \in W} \sum_{p \in P_w} x_p \ln \frac{x_p}{d_w}$$

have same solution with $\eta \to 0$. But the last task can be interpret as a game with logit-dynamics, which converges to single state with $\eta \to 0$. In such manner we found motivation for Bar-Gera selection.

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Covering a plane with equal sectors

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The problem of constructing the optimal cover of a plane with plane figures (disks, ellipses, sectors) is related to the computational geometry and arises in the context of different applications (such as wireless sensor networks). The cover of a plane region A is such set of figures C that each point in A belongs to at least one figure in C. One measure of efficiency is the coverage density, which is defined as the ratio of sum of the figures areas in the cover C to the area of region A.

The most studied problem is construction of the least dense covers of a plane with the disks of different radii [1, 2]. Cover of a plane domain can be considered fully specified if one knows the figures included in the cover, location of each figure and its orientation. If the figures in a cover are sectors, then it is necessary to know the radius, angle, the location of the vertex and the slope of the axis of each sector. Because of the variety of the covers, in the literature *regular* covers are often considered [2]. In the case of regular cover, plane is divided into regular polygons (tiles), and all the tiles are covered uniformly.

The aim of this work is construction of the new regular covers of the plane with *equal sectors*. It is still poorly studied problem and there are only a few publications on this topic [3, 4]. Naturally arises the problem of finding the plane cover with equal sectors in which the per unit area, the minimum number of sectors. Due to the technical reasons the parameters of the sector (angle and radius) can not take arbitrary values. If we take into account an additional cost of installation, it should be preferred embodiments, when several facilities are placed on the same site.

The foregoing consideration gives rise to the problem of constructing a regular plane covers with equal sectors with the minimum number of sectors per unit area. Additionally we require that each sector involved in covering of one tile, and the vertices of sectors covering one tile are located at one point. That is the problem we consider.

We first found the optimal models to cover one tile with equal disks depending on the angle of a sector. Based on these results we found the optimal solution for every angle (depending on the angle of the sector in the optimal cover, the triangular, square and hexagonal tiles can be used). The final theorem contains many statements and takes up much space. Here we present the theorem for

one type of a tile – regular triangle. Let $\sigma(\alpha)$ be the minimal number of sectors (with angle α) per unit area.

Theorem. *Let $\alpha \in [\alpha_1, \alpha_2]$, $0 < \alpha_1 < \alpha_2 \le \pi$ and $\alpha R^2 = S = const$. Then*

1. *if $[\alpha_1, \alpha_2] \subseteq (0, \pi/9]$, then $\sigma(\alpha) = \frac{2\alpha}{S\sqrt{3}} \min\{\lfloor \frac{\pi}{3\alpha} \rfloor / (1 - \cos(\lfloor \frac{\pi}{3\alpha} \rfloor \alpha)), 2\lceil \frac{\pi}{3\alpha} \rceil\}$;*
2. *if $[\alpha_1, \alpha_2] \subseteq [\pi/9, \pi/3]$, then $\sigma(\alpha) = \frac{2\alpha}{S\sqrt{3}} \min\{\lfloor \frac{\pi}{3\alpha} \rfloor / (1 - \cos(\lfloor \frac{\pi}{3\alpha} \rfloor \alpha)), 2\lceil \frac{\pi}{3\alpha} \rceil, 1/(1 - \cos\alpha)\}$;*
3. *if $[\alpha_1, \alpha_2] \subseteq [\pi/3, 5\pi/6]$, then $\sigma(\alpha) = 1/S_3(\alpha) = 4\alpha/(S\sqrt{3})$;*
4. *if $[\alpha_1, \alpha_2] \subseteq [5\pi/6, \pi]$, then $\sigma(\alpha) = 1/S_4(\alpha) = \alpha(1 + \sqrt{3}\tan\frac{\alpha}{2})^2 / (S\sqrt{3}\tan^2\frac{\alpha}{2})$.*

Main result is that to find the optimal coverage it is sufficient to compare a finite number of options, and the complexity of constructing the solution is $O(1)$.

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Piecewise quadratic functions application in linear optimization problems

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The report considers the issue of piecewise quadratic functions application in linear optimization problems. The functions concerned can be used in such fields as finding projection of a given point on the set of solutions of linear equation systems and/or inequalities (using the theorems of alternatives), constructing hyperplanes family separating polyhedra, determining the steepest descent for convex programming problems methods, svm-recognition methods, determining the projection of a given point on the set of solutions of linear programming (LP), etc. The piecewise quadratic function effectiveness is chiefly concerned with the generalized Newton method applied for their unconstrained minimization.

Several parallel versions of the generalized Newton method were implemented for unconstrained minimization of convex piecewise quadratic functions based on various optimization schemes of the original matrix data decomposition (column, row, and cellular schemes). The resulting parallel algorithms were successfully used to solve large-scale linear optimization problems (up to several dozen millions of variables and several hundred thousands of linear equations). Some computational experiments were performed on a cluster consisting of two-processor nodes based on 1.6 GHz Intel Itanium 2 processors connected by Myrinet 2000. For example, a cellular scheme integrating 144 processors within a cluster applied to solve an LP problem with one million variables and 10000 constraints accelerated the computation rate by approximately 50 times, with computation time 28 s. LP problem with two million variables and 200000 constraints was solved on 80 processors in about 40 min. Another LP problem with 60 million variables and 4 thousand constraints was solved on 128 processors by means of a column scheme in 140 s.

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Effective hull and its approximation

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This talk presents the concept of the *effective hull of a set* and the method of its approximation for the case when the set is an image of a compact set.

Let Λ_m denote the set of tuples of length m whose components are 1 or -1. Denotation $y_2 \succcurlyeq_\lambda y_1$ where $\lambda \in \Lambda_m$ means that the following m conditions are simultaneously valid:

$$\left(y_2^{(i)} - y_1^{(i)}\right) \lambda^{(i)} \geq 0 \text{ for } i = 1, \ldots, m. \tag{1}$$

If neither $y_1 \succcurlyeq_\lambda y_2$ nor $y_2 \succcurlyeq_\lambda y_1$ are valid vectors y_1 and y_2 are called *non-comparable for* $\lambda \in \Lambda_m$.

For $y \in \mathbb{R}^m$ and $\lambda \in \Lambda_m$ define sets $D_\lambda(y) = \{z \in \mathbb{R}^m : z \succcurlyeq_\lambda y\}$ and $D^\star_\lambda(y) = \{z \in \mathbb{R}^m : y \succcurlyeq_\lambda z\}$. These notations can be generalized for sets: $Z \subseteq \mathbb{R}^m$: $D_\lambda(Z) = \cup_{z \in Z} D_\lambda(z)$, $D^\star_\lambda(Z) = \cup_{z \in Z} D^\star_\lambda(z)$.

Let $Y \subseteq \mathbb{R}^m$ be a compact set. A point $y \in Y$ is called λ-*point*, if there are no such points $z \in Y$ that $z \succcurlyeq_\lambda y$ and $z \neq y$. The union of all λ-points is called λ-*border of a set* Y and is denoted by $\mathcal{P}_\lambda(Y)$, i.e.

$$\mathcal{P}_\lambda(Y) = \{y \in Y : Y \cap D_\lambda(y) = y\}. \tag{2}$$

The union of λ-borders $\mathcal{P}_{eff}(Y) = \cup_{\lambda \in \Lambda_m} \mathcal{P}_\lambda(Y)$ is called *effective border of* Y.

Statement 1. *The effective border of a strictly convex compact set coincides with its border.*

For arbitray convex and non-convex sets this theorem is not valid.

Set $H(Y) = \cap_{\lambda \in \Lambda_m} D^\star_\lambda(\mathcal{P}_\lambda(Y))$ is called *effective hull of* Y.

Statement 2. *For a compact set* Y *the following inclusion is valid* $Y \subseteq H(Y) \subseteq Conv(Y)$.

Let $\varepsilon \geq 0$ and $\lambda \in \Lambda_m$. A set Y^λ_ε from $\mathbb{R}^m$ is called ε-*approximation of the* λ-*border*, if for any $y_* \in \mathcal{P}_\lambda(Y)$ there exists a point y_ε from Y^λ_ε such that $y_\varepsilon + \varepsilon\lambda \succcurlyeq_\lambda y_*$ and the set Y^λ_ε contains only non-comparable points.

Consider a collection $\{Y^\varepsilon_\lambda\}, \lambda \in \Lambda_m$ of ε-approximations of the λ-borders. The $\cup_{\lambda \in \Lambda_m} Y^\varepsilon_\lambda$ is called ε-*effective border* of Y. *External* ε-*effective border* is a set $\cup_{\lambda \in \Lambda_m} (Y^\varepsilon_\lambda + \varepsilon\lambda)$.

A set $\cap_{\lambda\in\Lambda_m} D^\star_\lambda\,(Y^\varepsilon_\lambda + \lambda\varepsilon)$ is called *ε-effective hull of Y*. The following statement is valid

Statement 3 *Let $H_\varepsilon \cap_{\lambda\in\Lambda_m} D^\star_\lambda\,(Y^\varepsilon_\lambda + \lambda\varepsilon)$ be ε-effective hull of Y, and $\overline{B}_\varepsilon = \cup_{\lambda\in\Lambda_m}\,(Y^\varepsilon_\lambda + \lambda\varepsilon)$ be a respective ε-effective border. Then the following inclusion is valid*

$$Y \subseteq H_\varepsilon \subseteq Conv(\overline{B}_\varepsilon). \tag{3}$$

This statement provides an efficient way to construct approximations of the convex hull of a set Y: first external ε-effective border is constructed which is a finite set. Then the convex hull of this border is obtained by existing and well-known algorithms.

We developed a deterministic algorithm for constructing ε-effective hull of Y when $Y = F(X)$, where X is a compact set and $F : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous mapping. This algorithm is based on non-uniform covering method proposed in [1]. In the talk we outline an important application of the proposed approach for describing working set of robotic manipulators.

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Reachable set approximation algorithms for smooth and discontinuous systems

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The authors propose a number of methods to obtain external and internal estimates, as well as approximations of the boundary of the reachable set of continuous controled system. The main difficulty, that does not allow directly apply previously developed methods for discontinuous systems, is the integration which must be carried out quickly enough, as is repeated many times. We suggest several methods of integration and compare the results of its application, that allows to make a conclusions about the correctness of the received approximations. They are all based on the implicit Euler's method with the conversion, the simplest version that do not use an event function are methods with fixed small step and with adjusts not fixed step. Also the method with changing step is implemented, it avoids the accumulation of errors by taking into account the value of the predicate characterizing the gap.

Realizations of integration allows us to apply the method of stochastic approximation, which is the most simple and reliable method giving an internal evaluation, as well as it arise to be possible to adapt the developed by the authors algorithms of uniform and a quasiuniform approximation of reachable set. These algorithms requires multiple solution of auxiliary optimization problems for adding points to the approximating set and thus, unlike the method of stochastic approximation resulting in a cloud of points uniformly (with some accuracy) approximating the set even with a small number of points. There was implemented several variants of algorithms with different statement of auxiliary optimization problems, the question of parameters settings was investigated in each case. We conducted computational experiments for smooth and discontinuous dynamical systems, which confirmed the practicable computational efficiency of the proposed approaches and allowed us to estimate the scope of the implemented algorithms

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Packing regular polygons into a bounded domain

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The article presents linear models for an approximate solution of the problem of packing regular convex polygons into a given closed bounded domain G, where the total area of the packed figures takes its maximal value.

The problem of packing rectangles and, in particular, squares of various sizes is studied in many papers, see reviews Lodi, Martello and Monaci [1], Co-Qman, Garey, Johnson [2]. There many models and solution methods for these problems exist. In many cases, one reduces the packing problem of rectangles to a linear programming problem; see Gilmore, Gomery [3]. One can solve the packing problem of rectangles and squares using heuristic techniques, genetic algorithms, artificial neuron networks, approximation algorithms and many other methods.

Let $P_{ka}(s)$ denotes the open regular k-gon with the side length of a and the center at the point s. We consider the problem of packing regular polygons into a set G for the following cases: one has to pack as many equal regular polygons $P_{ka}(s)$ (with known k and a) into the set G as possible and to define locations of their centers; one has to pack into G regular polygons $P_{ka}(s)$ and (or) $P_{kb}(s)$ ($a \neq b$) with known values of k, a and b so as to maximize the total area of packed polygons and to define locations of their centers. We consider the packing of polygons without their rotation.

Let us place the polygon $P_{kb}(s)$ with respect to fixed $P_{ka}(O)$ in all possible ways (without rotation about the center s). In this case all possible locations of the point s in the plane form a closed curve; we denote it by $M_{k-a-b}(O)$, where O is the center of $M_{k-a-b}(O)$. When packing polygons $P_{ka}(O)$ and $P_{kb}(s)$, one should make the center of the polygon $P_{kb}(s)$ lie outside $int(M_{k-a-b}(O))$. In order to fulfill this condition, we use block norms introduced by Ward and Wendell [5] or some other conditions ensuring that a point belongs or do not belongs to a polygon.

In this work, using results by Galiev and Lisafina [4], the linear models for an approximate solution of packing regular polygons are presented. We construct grids in G; the nodes of these grids form a finite set of points W, and it is assumed that the centers of figures to be packed can be placed only at the points of W. The problems of packed figures with the centers at the points of

W are reduced to 0-1 linear programming problems. A heuristic algorithm for solving the packing problems based on linear models is proposed. For calculating coefficients of the constraint matrix and for solving the packing problem for polygons we have developed a computer program based on the optimization software package CPLEX 11.2.

Numerical results demonstrate the effectiveness (performance) of this approach.

The proposed approach is characterized by the following important advantages: (a) allows to find the packing for an arbitrary connected bounded closed domain in a unified manner, (b) if the packing is found without heuristic, then the global maximum of the total area of the packed figures (with the centers in the grid nodes) is provided, (c) allows to take into account restrictions on the number of the figures of the given sizes.

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Multiple coverings of triangle, square and circle by circles

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A set of n closed circles K_j $(1{\leq}j{\leq}n)$ of the equal radius r forms a k-fold $(1{\leq}k{\leq}n)$ covering of G if every point s of a set G belongs to at least k of those circles. We state the following problem.

$Z1$. Find an arrangement of n circles $(1{\leq}j{\leq}n)$ of the equal radius r forming a k-fold covering of G such that their radius r has the minimum possible value $r^*_{n,k}$.

Problems of covering a bounded part of the plane have many applications in locating various stations (base stations for cellular networks, wireless Internet, ambulance stations, etc.). Surveys of various applications and methods for solving covering problems can be found, for example, in [1, 2]. The design of navigation systems (for example, global positioning systems such as NAVSTAR, GLONASS, and Galileo) also involve multiple covering problems.

In the available literature covering problems for various plane figures (square, rectangle, triangle, and circle) are studied. Multiple coverings of a bounded domain and, in particular, of a square are studied in [3]. However, we are not aware of studies devoted to multiple coverings of triangles and circles.

Let $c_j(x_j, y_j)$ be the centers of circles K_j $(1{\leq}j{\leq}n)$. Let $\xi = (x_1, y_1, ..., x_n, y_n)$, and let $d(s,t)$ be the Euclidean distance between the points s and t on the plane P. It is well known that k-fold covering $(k{\geq}1)$ problem is reduced to the problem $\min\limits_{\xi} \max\limits_{s\in G} \min\limits_{j\in J_k} d(s, c_j)$, where J_k is the set of indices of $n-k+1$ centers c_j that are not closer to s than the other $k-1$ centers c_j $(1{\leq}j{\leq}n)$.

The well-known set covering problem can be formulated as the integer programming problem:

$Z2$: $min\{cz : Az{\geq}b, z_i{\in}\{0,1\}\}$ or $min\{cz : Az{\geq}b, z_i{\in}Z^+\}$.

Here, cz is the scalar product of the vectors c and z, $c = (c_1, ..., c_m)$, $z = (z_1, ..., z_m)^T$, A is an $l{\times}m$-matrix with the elements equal to 0 or 1, $b = (b_1, b_2, ..., b_l)^T$ is a given l-dimensional vector, and Z^+ is the set of non-negative integers. If $b_1 = b_2 = ... = b_l = 1$, then this problem is the set covering problem; in the case $b_1 = b_2 = ... = b_l = k(k > 1)$, it is a multiple set covering

problem. In paper, the problem $Z2$ is considered for a special case of the matrix of constraints A - it is a square matrix constructed based on covering a finite set with circles of a given radius.

The problem $Z1$ is a continuous covering problem, while $Z2$ under the constraints on the matrix A specified above we called a discrete covering problem.

The procedure of solving the original problem $Z1$ is to consistently solve discrete covering problem and continuous k-fold covering problem. The method based on the use of the discrete approach (grid construction) and the continuous approach is fairly universal; it makes it possible to solve different problems, for example, covering and p-median problems.

Algorithms for solving multiple covering problems of the equilateral triangle, square, and circle with a given number of congruent circles of the minimum possible radius are proposed. Optimality of the constructed coverings for certain n and k ($k \leq n$) is proved. For $n \leq 15$ and $1 < k \leq n$, numerical methods are used to estimate the radius $r_{n,k}$ of circles for which the required k-fold covering of the equilateral triangle, square, and circle can be ensured.

The algorithms presented in this paper make it possible to find multiple coverings not only for the equilateral triangle, square, and circle, but also for arbitrary bounded domains in the plane.

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A new method of finding the projection of a point on the solution set of a primal linear programming problem

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Let the primal LP problem be presented in the canonical form:

$$f_* = \min_{x \in X} \ c^\top x, \quad X = \{x \in R^n \ : \ Ax = b, \ x \geq 0_n\}. \tag{P}$$

We can rewrite (P) in the following equivalent form:

$$f_* = \min_{x, x_0 \in Z} \ x_0, \tag{P'}$$

$$Z = \{x \in R^n, x_0 \in R^1 \ : \ Ax = b, \ -c^\top x + x_0 = 0, \ x \geq 0_n\}.$$

The dual problem obtains the form:

$$f_* = \max_{u, u_0 \in U} \ b^\top u, \quad U = \{u \in R^m \ : \ A^\top u - cu_0 \leq c \ , u_0 = 1\}. \tag{D'}$$

Let $\hat{z}$ be a fixed arbitrary vector in R^{n+1}, $\hat{z}^\top = [\hat{x}^\top, \hat{x}_0]$. As it was shown in [1-4], the problems $(P'), (D')$ reduce to the following unconstrained maximization problem:

$$I = \max_{p \in R^m, p_0 \in R^1} \ S(p, \beta, \hat{z}),$$

where β is a fixed scalar, and the function $S(p, \beta, \hat{z})$ is determined by:

$$S(p, \beta, \hat{x}) = b^\top p - \frac{1}{2} \|(\hat{x} + A^\top p - \ p_0 c)_+\|^2 - \frac{1}{2} \ (\hat{x}_0 + \ p_0 - \beta)^2.$$

Also, there exists β_* such that for all $\beta \geq \beta_*$ the solution $p(\beta), \ p_0(\beta)$ to problem (1) define the projection of the point $\hat{z}$ on the solution set of the problem (P') by formulas:

$$\begin{bmatrix} \hat{x}^* \\ \hat{x}_0^* \end{bmatrix} = \begin{bmatrix} (\hat{x} + A^\top p(\beta) - \ p_0(\beta) c)_+ \\ \hat{x}_0 + \ p_0(\beta) - \beta \end{bmatrix}.$$

There is an important difference between unconstrained maximization problem (1) and corresponding unconstrained maximization problem in [1-4]. In the first case the parameter β appears only in a single component of the maximized function S.

In this work the new threshold value β_* has been estimated and numerical experiments with large-scale LP problems have been carried out in order to compare their results with those obtained in [2]. The generalized Newton method [5] with a stepsize chosen Armijo's rule was used for solving unconstrained maximization problem (1).

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Iterative solver for stiff variational problem related to construction of thick new-wall mesh layers and offsets

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Thick near-wall prismatic layers and large offsets can be constructed using springback procedure when thin highly compressed layer of polyconvex hyperelastic material is attached to fixed surface and is allowed to expand. This procedure allows to construct one-cell-wide layers and offsets with thickness comparable to the characteristic size of the body. Resulting mesh layer does not contain inverted cells. Self-intersection zones can be easily eliminated by cutting off excessive thickness. This procedure generally does not lead to offset thickness reduction due to local dents or elevations which is quite different from the advancing front collision detection technique. When small surface elements are present discrete variational problem can become quite stiff due to very large target height-to-base ratio of expanding elasting cells. Problem with convergence of iterative procedure is manifested as final thickness falling short of target values. Thus iterative minimization of hyperelastic stored energy needs finely tuned preconditioning. Note that the Hessian matrix for discrete polyconvex stored energy is not positive definite and problem of choosing good positive definite preconditioner is not trivial. We present implicit solver for variational springback problem and illustrate its behaviour on hard real-life test cases.

Construction of prismatic layer for relatively realistic airplane test model is illustrated in Fig.1. Consecutive steps are shown, such as one-element-wide layer construction, self-intersection elimination, Laplace-Beltrami smoothing and layer subdivision with inner layers orthogonalization.

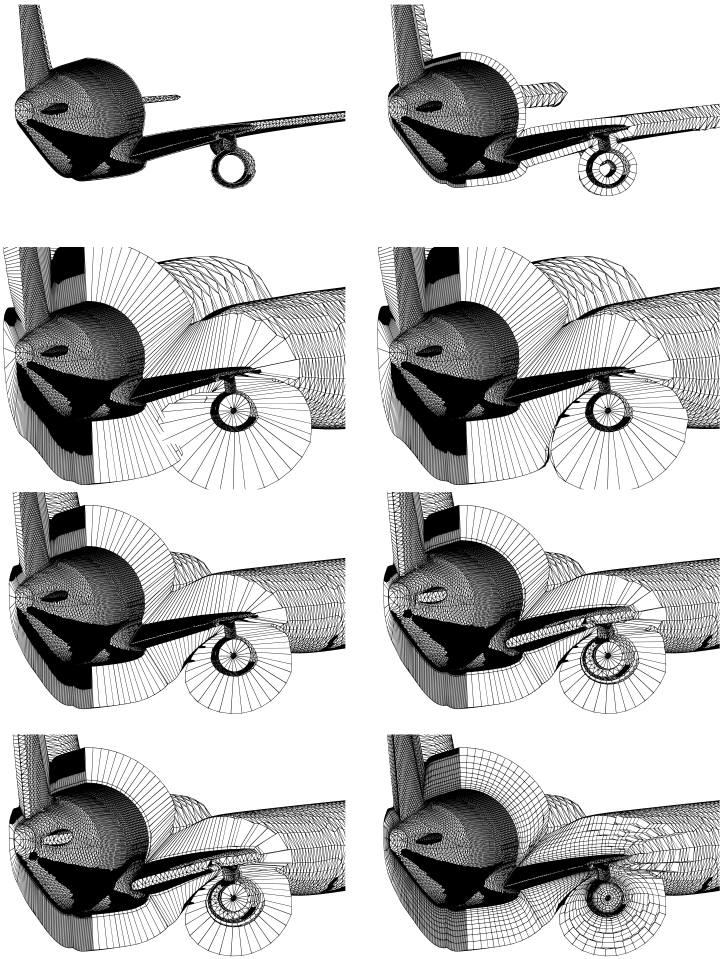

Fig. 1

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Efficient adaptive algorithm for solving ELP problems

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Consider the following maximum entropy linear programming (ELP) problem on simplex:

$$\begin{cases} f(x) = \sum_{j=1}^m x_j \ln x_j + \sum_{j=1}^m c_j x_j \to \min\limits_{x \in Q} \\ Q = \{x \geq 0 : \sum_{j=1}^m x_j = 1\} \\ \sum_{j=1}^n a_{ij} x_j = b_j, i = 1, \ldots, n \end{cases}$$

This problem arises in various applications, including Traffic Rank model, transport models, chemical kinetic equilibrium problems.
A dual problem can be written as follows:

$$\phi(\lambda) = \sum_{i=1}^n \lambda_i b_i + \ln \left(\sum_{j=1}^m \exp(-c_j - 1 - \sum_{i=1}^n \lambda_i a_{ij}) \right) \to \max_{\lambda}.$$

Given the solution of the dual problem, the solution to primal problem can be explicitly found. The complication is that the residual of the solution of dual problem ($\lambda^\star$) cannot be controlled, because dual problem has no equality constraints on Lagrange multipliers (λ). In order so solve this it is proposed to l_2-regularize dual problem and then solve it with modified Fast Gradient Method:

$$\phi_\delta(\lambda) = \phi(\lambda) - \frac{\delta}{2}\|\lambda\|_2^2, \quad \delta > 0.$$

We're looking for a solution that for given $\epsilon_f, \epsilon > 0$ will satisfy the following:

$$x(\lambda^*) \in Q, \quad \|Ax(\lambda^*) - b\|_2 \leq \epsilon, \quad f(x(\lambda^*)) \leq f^* + \epsilon_f$$

where $f^* = \min\limits_{x \in Q} f(x)$.

Iterative FGM process looks as follows:

$$\begin{aligned} \lambda_{k+1} &= u_k + \tfrac{1}{L_\phi + \delta} \nabla \phi_\delta(u_k) \\ u_{k+1} &= \lambda_{k+1} + \tau(\lambda_{k+1} - \lambda_k), \quad k \geq 0. \end{aligned}$$

where $\tau = \frac{[L_\phi + \delta]^{1/2} - \delta^{1/2}}{[L_\phi + \delta]^{1/2} + \delta^{1/2}}$ and $\lambda_0 = 0$ and $u_0 = 0$

$L_\phi = \frac{1}{\sigma(f)}\|A\|^2 = \max\{(Ax,\lambda)\,\|x\|_1 \le 1, \|\lambda\|_2 \le 1\} = \max\limits_{1\le j\le m}\|A^{\langle j\rangle}\|_2.$

Where strictly convexity constant $\sigma(f)$ in 1-norm is equal 1 for the entropy function f.

O. Devolder et al. discuss double smoothing technique, proposing to apply FGM to the sequence of regularized dual problems, gradually reducing smoothness parameter δ.

Algorithm:

Initial step: choose $\epsilon > 0$, initial $\delta(1) = \frac{\varepsilon}{8}$, $\lambda_0 = 0$ and $u_0 = 0$.

Step j:

1. Apply FGM to the regularized problem $\varphi_{\delta(j)}(\lambda)$ with stopping criterion $\left\|\nabla\varphi_{\delta(j)}(\lambda_k)\right\|_2 \le \frac{\varepsilon}{4}$.
2. If the condition $\|\nabla\varphi(\lambda_k)\|_2 \le \varepsilon$ holds true, then dual solution is found and primal solution can be calculated.
3. Otherwise if $\|\nabla\varphi(\lambda_k)\|_2 \le \varepsilon$ condition doesn't hold, then $\delta(j+1) = \delta(j)/2$ and $j = j+1$. Goto 1.

Number of j steps is estimated as $O\left(\max\left(1, \log_2\|\lambda^*\|_2\right)\right)$.

This algorithm gives the following estimation of number of iterations:

$$\mathrm{O}\left(\ln\left(\|\lambda^*\|_2\right)\sqrt{\frac{\|\lambda^*\|_2 \max\limits_{l=1,\dots,m}\|A^{\langle l\rangle}\|_2}{\varepsilon}}\ \ln\left(\frac{\|\lambda^*\|_2 \max\limits_{l=1,\dots,m}\left\|A^{\langle l\rangle}\right\|_2}{\varepsilon_f\varepsilon^2}\right)\right)$$

This estimation depends on the dual solution $\|\lambda^*\|_2$. Although the dual problem solution can be estimated with an upper bound of $O(\frac{\ln n}{\epsilon})$.

Related references can be found at `http://arxiv.org/abs/1405.7630` (in Russian).

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On asymptotical solvability for some problems of finding several disjoint Hamiltonian cycles in a weighted complete graph

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We consider the m-Peripatetic Salesman Problem (m-PSP) that is a natural generalization of the classical Traveling Salesman Problem (TSP). Given a complete directed n-vertex graph $G = (V, E)$ with weight functions $w_i : E \to R^+,\ i = 1, \ldots, m$, the problem is to find m edge-disjoint Hamiltonian circuits $H_1, \ldots, H_m \subset E$, such that minimize their total weight:

$$\sum_{i=1}^{m} w_i(H_i) = \sum_{i=1}^{m} \sum_{e \in H_i} w_i(e).$$

The problem is investigated both for deterministic or random instances, for arbitrary and metric weight functions which can be different or common for all circuits and for special classes of graphs with edge weights belonging to a given interval or a finite set of numbers.

Applications include the design of watchman tours where it is often important to assign a set of edge-disjoint rounds to the watchman in order to avoid always repeating the same tour and thus enhance security. It is known a network design application where, in order to protect the network from link failure, several edges-disjoint cycles must be determined. There is a scheduling application of the 2-PSP where each job must be processed twice by the same machine but technological constraints prevent the repetition of identical job sequences. All nontrivial variants of TSP and m-PSP are NP-hard.

In this report we present new polynomial algorithms for the minimum-weight m-PSP on directed or undirected graph with different weight functions of edges in Hamiltonian circuits on random instances.

We say that a distribution function $\mathcal{F}'(x)$ is called $\mathcal{F}$-majorizing if it dominates the function $\mathcal{F}(x)$: $\mathcal{F}'(x) \geq \mathcal{F}(x)$ for every x. This kind of functions we shall say as the $\mathcal{F}$-majorizing type.

The case is considered when the entries of the distance matrix are some independent identically distributed random variables with the values in the bounded range $[a_n, b_n]$ [1], and the range $[a_n, \infty]$, unbounded from above, $a_n > 0$

[2]. First class instances is given for the distribution function of UNI$[a_n, b_n]$-majorizing type, the second class has $\mathcal{F}_\beta$-majorizing type, where UNI$[a_n, b_n]$ is uniform distribution in the interval $[a_n, b_n]$, and $\mathcal{F}_\beta(x)$ is exponential distribution with a parameter $\beta = \beta_n$:

$$\mathcal{F}_\beta(x) = 1 - \exp\left(\frac{x - a_n}{\beta}\right),\ x \geq a_n.$$

We construct efficient algorithms with time complexity $O(mn^2)$, obtain the performance guarantees for the relative errors and the failure probabilities for random instances of UNI$[a_n, b_n]$-majorizing type and of $\mathcal{F}_\beta$-majorizing type. Particularly, from the inequality

$$\int_0^x \frac{2}{\sqrt{\pi}} \exp\left(-\frac{u^2}{2}\right) du \geq 1 - \exp\left(-\frac{x}{2}\right)$$

it follows that the obtained results are true for the m-PSP with truncated normal distribution with a parameter σ_n.

The conditions of asymptotic optimality of the algorithms established in the form

$$\beta/a_n = O(n^\theta) \text{ and } m \leq n^{1-\theta},$$

where β is equal to one of corresponding parameter b_n, β_n or σ_n; $\theta < 1$.

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Efficient approximation algorithms for some problems of finding several disjoint cliques in a weighted complete undirected graph

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We investigate discrete optimization problems induced, in particular, by important problems of data analysis and pattern recognition. Our aim is analyzing the computational complexity of these problems, constructing efficient algorithms, and estimating their performance guarantees. Earlier it was proved that the problem of selecting from a set of Euclidean vectors a subset of given size minimizing the sum of squares of all possible pairwise distances between its elements is NP-hard in the strong sense [1]. This statement is one of the possible formalizations of the cluster analysis problem, which is very important in data analysis. This problem is essentially about grouping similar objects.

In [2] the Weighted Clique Problem (WCP) for finding a minimum clique (with respect to the total weight of its vertices and edges) of fixed size in a complete undirected weighted graph was considered. The inapproximability of the WCP is established for the general case and a 2-approximation efficient algorithm with time complexity $O(n^2)$ is proposed for the cases where the weights of vertices are nonnegative and the weights of edges either satisfy the triangle inequality or are squared pairwise distances for some system of points of the Euclidean space.

Now we consider the following generalization of the WCP:

m-Weighted Clique Problem (m-WCP).

Input: a complete undirected graph G with nonnegative weights of edges and vertices, and positive integers $L_1, \ldots, L_m$ such that $\sum_{i=1}^{m} L_i \leqslant n$.

Output: a family of disjoint cliques in G of orders $L_1, \ldots, L_m$ with the minimal total weight of vertices and edges contained in these cliques.

We show that the m-WCP is NP-hard in a strong sense and inapproximable in the general case and in some of its important special cases. We propose an

approximation algorithm with time complexity $O(n^{m+2}\log n)$ in the case when the weights of the vertices of the graph G are nonnegative and the weights of the edges either satisfy the triangle inequality or are squared pairwise distances between some points of the Euclidean space.

The idea of approximation for m-WCP is based on the use of the exact solution of the special (auxiliary) problem of finding m disjoint stars of corresponding orders. It is shown that the exact solution of this problem can be found in time $O(n^{m+2}\log n)$ using the transportation algorithm. This algorithm is polynomial if m is fixed. The proposed approximation algorithm for the problem m-WCP has the same time complexity. It is proved that for the Metric m-WCP we have the approximation guarantee

$$2\Big(1-\frac{\sum_{i=1}^{m} S(B_i^*)}{\sum_{i=1}^{m} L_i S(B_i^*)}\Big),$$

where $S(B_i^*)$ is the total weight of the vertices and the edges in the i-th star of the auxiliary problem, $i=1,\ldots,m$. For the Quadratic Euclidean m-WCP the approximation ratio 2 is proved.

So the present paper generalizes and deepens the results obtained in [2]. We hope that the proposed approach can be extended for other actual subclasses of m-WCP.

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On an Approximate Solution Method of a Finite Three-Player Game

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A non-cooperative three-player game Γ is defined by the players' sets of strategies X_1, X_2, X_3 and their payoff functions f_1, f_2, f_3. Introduce the function

$$N(x) = N(x_1, x_2, x_3) = \sum_{i=1}^{3} \max_{x_i \in X_i} f_i(x_1, x_2, x_3) - f(x_1, x_2, x_3),$$

$x = (x_1, x_2, x_3)$, defined over the set $X = X_1 \times X_2 \times X_3$, where $f(x) = \sum_{i=1}^{3} f_i(x)$. Since $N(x) \geq 0$, $x \in X$, and $N(x) = 0$ if, and only if x is a Nash point of the game Γ, then solving the game Γ is tantamount to finding a global minimum of the function $N(x)$ on X. In order to describe a finite game Γ, it is convenient to make use of the tables whose entries are numbered with the indices s_1, s_2, s_3 running from 1 to n_1, n_2, n_3, respectively; here n_i denotes the number of distinct strategies of player i, $i = 1, 2, 3$. Now consider the finite three-player game Γ with the payoff values of player i defined by the table $A_i = (a^{(i)}_{s_1 s_2 s_3})$, where $a^{(i)}_{s_1 s_2 s_3}$ is his/her payoff given that player α has selected the strategy s_α, $\alpha = 1, 2, 3$. By introducing the players' mixed strategies, we come to the game Γ, where $X_i = \{x_i = (x_{i1}, \ldots, x_{in_i})\colon \sum_{s_i=1}^{n_i} x_{is_i} = 1,\ x_{is_i} \geq 0,\ 1 \leq s_i \leq n_i\}$, $f_i(x) = \sum_{s_1=1}^{n_1} \sum_{s_2=1}^{n_2} \sum_{s_3=1}^{n_3} a^{(i)}_{s_1 s_2 s_3} x_{1s_1} x_{2s_2} x_{3s_3}$, $i = 1, 2, 3$, $x = (x_1, x_2, x_3)$.

Although the problem of global minimization of the function $N(x)$ by $x \in X$ is extremely difficult owing to the existence of its numerous local minima apart from the global one, the problem of minimization of this function with respect to *solely one* of the three vector variables composing x, can be easily reduced to a linear program.

Indeed, fix the variables $x_2 \in X_2$, $x_3 \in X_3$ and set for $1 \leq s_i \leq n_i$, $i = 1, 2, 3$, $a^{(2)}_{s_1 s_2}(x_3) = \sum_{s_3=1}^{n_3} a^{(2)}_{s_1 s_2 s_3} x_{3s_3}$, $a^{(3)}_{s_1 s_3}(x_2) = \sum_{s_2=1}^{n_2} a^{(3)}_{s_1 s_2 s_3} x_{2s_2}$, $a_{s_1}(x_2, x_3) = \sum_{s_2=1}^{n_2} \sum_{s_3=1}^{n_3} a_{s_1 s_2 s_3} x_{2s_2} x_{3s_3}$, $a_{s_1 s_2 s_3} = \sum_{i=1}^{3} a^{(i)}_{s_1 s_2 s_3}$. Then the problem of minimization of the function $N(x)$ by $x_1 \in X_1$ is equivalent to the linear program $P_1(x_2, x_3)$ defined as follows:

$v_{11} + v_{21} - \sum_{s_1=1}^{n_1} a_{s_1}(x_2, x_3) x_{1s_1} \to \min$, $\sum_{s_1=1}^{n_1} a^{(2)}_{s_1 s_2}(x_3) x_{1s_1} \leq v_{11}$, $1 \leq s_2 \leq n_2$, $\sum_{s_1=1}^{n_1} a^{(3)}_{s_1 s_3}(x_2) x_{1s_1} \leq v_{21}$, $1 \leq s_3 \leq n_3$, $\sum_{s_1=1}^{n_1} x_{1s_1} = 1$, $x_{1s_1} \geq 0$, $1 \leq s_1 \leq n_1$.

In a similar manner, the problems of minimization of $N(x)$ with respect to $x_2 \in X_2$ or $x_3 \in X_3$ are reduced to the linear programming problems $P_2(x_1, x_3)$

or $P_3(x_1, x_2)$, respectively.

Let X_i^0 be the set of pure strategies of player i, $i = 1, 2, 3$. Fix an arbitrary pair of pure strategies $x_2 \in X_2^0$, $x_3 \in X_3^0$ and determine a sequence of points $x^k = (x_1^k, x_2^k, x_3^k)$ as follows: x_1^1 is a solution of problem $P_1(x_2, x_3)$, $x_2^1 = x_2$, $x_3^1 = x_3$, x_1^{k+1} is a solution to problem $P_1(x_2^k, x_3^k)$, x_2^{k+1} is a solution of $P_2(x_1^{k+1}, x_3^k)$, x_3^{k+1} is a solution to $P_3(x_1^{k+1}, x_2^{k+1})$, $k = 1, 2, \ldots$; denote by $q_1 = q_1(x_2, x_3)$ the limit of the monotone decreasing sequence of $N(x^k)$. Similarly, generate two more sequences of the triples of strategies for the two other players and determine the corresponding functions $q_2 = q_2(x_1, x_3)$ and $q_3 = q_3(x_1, x_2)$ for the fixed $x_1 \in X_1^0$, $x_3 \in X_3^0$ or $x_1 \in X_1^0$, $x_2 \in X_2^0$, respectively.

The proposed method of solution of the three-player game Γ is based upon an approximate calculation of the values of functions q_i, $i = 1, 2, 3$. Namely, as such an approximate value $\widetilde{q}_i$, we select $N(x^k)$ whenever the difference, $N(x^k) - N(x^{k+1})$, drops below a preliminarily fixed tolerance parameter $\varepsilon > 0$.

When solving the game Γ by the developed method, altogether $n_2 n_3$ values of $\widetilde{q}_1(x_2, x_3)$, $n_1 n_3$ values of $\widetilde{q}_2(x_1, x_3)$, and $n_1 n_2$ values of $\widetilde{q}_3(x_1, x_2)$, are computed. The minimum of all these values is found, and the corresponding (mixed) strategies are reported as an approximate solution of game Γ. The total amount of linear programs to be solved in order to determine the approximate solution of game Γ equals $3(n_1 n_+ n_2 n_3 + n_1 n_3)l$, where l is the average number of the triples of linear programs required to solve in order to determine the approximate values of the functions q_i. Based upon our experience of making use of an analogous method applied to the solution of bi-matrix games, we could predict that the value of l would usually drop between 10 and 20.

On design of a vibrating beam on an elastic foundation for the maximum fundamental frequency

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This report is concerned with the study of optimization problems for the fundamental frequency of a vibrating beam on a continuous elastic foundation with various boundary conditions. During the last few decades, a lot of attention has been paid to optimal structural design for elastic bodies. Despite that there are many open problems in this field. For example, it is yet unknown whether numerical procedures proposed in many papers converge to an actual optimal solution. Furthermore, it is not established in most cases if necessary optimality conditions are also sufficient. Within the current study a strict proof of that has been obtained for some cases. In particular, it can be easily applied to the results of [3, 4].

A classical approach for such problems [1, 2] is to seek for a suitable cross-sectional area distribution that yields an optimum value of the fundamental frequency of a beam with a specified length and volume. Due to considerable achievements in composite material technology, it has also become possible to attain an increase in fundamental frequency by varying the material properties [3, 4]. In this study both of these approaches are under considerations.

For a given beam, free transverse vibrations are governed by the following differential equation

$$(p^{\alpha}u'')'' + \kappa u = \lambda_i p^i u, \quad \alpha > 0, \quad x \in (0,1). \tag{1}$$

Hereafter all the quantities are considered as dimensionless. The case $i = 1$ corresponds to a homogeneous beam with geometrically similar cross sections, $p(x)$ being the cross-sectional area distribution. For $i = 0$, in turn, a beam is regarded as uniform and non-homogeneous. In this case $p(x) = c\,e(x)^{\frac{1}{\alpha}}(x)$, where c is a constant, $e(x)$ is the non-dimensional Young's modulus of the beam material.

For a fixed p, let $\lambda_i^1(p)$ be the lowest eigenvalue of (1). The optimization problems under study may thus be presented as follows:

$$\max_p \lambda_i^1(p) = \max_p \min_v J_i(p,v), \quad \int_0^1 p(x)\,dx = 1.$$

Here $J_i(p, v)$ is the Rayleigh quotient for (1), $v(x)$ ranges over some set of admissable functions satisfying prescribed boundary conditions.

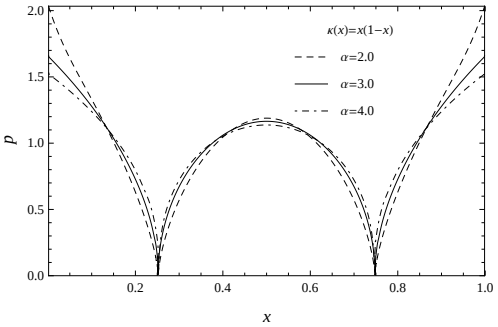

Fig. 1 Variation of p along the axis for clamped-clamped uniform beams.

As it turns out, exact analytical solutions can be derived in some cases. For the rest cases an efficient gradient-type method based on a necessary optimality condition was proposed. Various numerical tests were used to show up convergence features of the method. Some obtained numerical solutions are depicted in Figure 1.

Further investigations in this direction are on their way.

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Practical optimization for nonconvex optimal control problems

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Solving nonconvex optimal control problems (OCP) continues to be one of the most challenging issues in extremal problems theory today. The theoretical foundations for the search of global extremum objective functional in an infinite dimensional space are being developed at the current moment of time. Unfortunately, algorithms for computational solution of OCP's with nonconvex functionals based on dynamic programming do not allow one to address many problems that arise in practical applications due to the "curse of dimensionality" despite the efforts of many well-known scientists. The few attempts to apply genetic and other "bioinspired" algorithms for this class of problems also leave unanswered some fundamental questions about the status of the solutions obtained.

This report discusses several new algorithm assemblages for nonconvex OPCs which are heuristic in nature [1]. First, we have realized the natural approach based on the method of stochastic multistart, that allows us to obtain not only best of known solutions, but in many cases also estimate the attainable set of the dynamical system and the dimensions of attraction domains for different extrema [2]. Second, we have conducted numerical studies of sequential discretization techniques that allow us to use updated approaches developed in the global extremum theory for finite-dimensional problems to solve OCPs. A technology of obtaining the lower estimates for the objective functional has been developed based on R. Gamkrelidzes convexifying method. We have managed to realize several new algorithms assemblages using the existing experience in solving phase estimation problems and we have also theoretically investigated some properties of the attainable set, such as: "grids algorithm", "curvilinear search algorithm", tunneling algorithm [3–4], etc. The deterministic method based on a generalization of Pontryagin's maximum principle shows good computational characteristics for small dimension OPCs. The ideas of globalization proposed by professor K.L. Teo, the approximation concepts of terminal functions using Shepard's operator, the application of McCormick's convexifying technique, etc. have been implemented in a number of new algorithms.

On the basis of the proposed algorithms software has been developed for nonconvex OCPs that allows creating interactive multimethod computational

schemes, controlling the computing process, verifying the quality of the dynamical system discretization, and estimating the probability of achieving the global extremum. Also parallel versions of a number of algorithms have been realized. A collection of nonlinear nonconvex OCPs comprising more than 100 model and content test problems has been assembled [5]. The properties of the implemented algorithms and their combinations have been evaluated in the course of several multivariate computational experiments. With the help of this software a number of applied OCPs from different areas have been solved [1], [6]: mechanics, cosmic navigation, flight dynamics, robotics, electrical energy, ecology, economics, medicine, geography, and nanophysics.

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Minimizing Lennard-Jones Function via Fractional Programming

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As well known [1] the problem of minimizing the potential energy of molecules is a nonconvex optimization problem with a large number of variables and complex nonlinear structures. As a consequence, a direct application of classical optimization methods and applied programs packages for solving nonconvex problems are known to be inefficient.

We proposed to approximate the problem of minimizing the Lennard-Jones potential energy of the molecule [1] by a special fractional optimization problem. The difficulty of the approximating problem is generated by nonsmoothness and nonconvexity of the objective function. We have proved that it is possible to reduce the fractional programming problem to the equation with the optimal value of the nonconvex optimization problem. This result provides a theoretical basis for developing numerical methods for the approximating problem.

We constructed a decomposition of nonconvex goal function as a difference of two convex functions and applied the Global Search (GS) Theory [2]. The classical nonsmooth optimization methods [3] are employed in the interior of the general GS scheme. The developed algorithm was exploited for solving the problem under scrutiny. Computational experiment was conducted on the examples from the Cambridge Cluster Database.

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Identification of the Hydraulic Resistance Coefficient

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In the work, we consider an approach to the identification of the hydraulic resistance coefficient of a linear section of a long-distance oil-pipeline when transporting hydrocarbon raw materials. The considered identification problem is reduced to the class of finite-dimensional optimization problems. To solve the derived optimization problem, we propose to use efficient numerical methods of first-order finite-dimensional optimization. With this purpose, in the work, we derive formulas for the components of the gradient of the objective functional in the space of identifiable parameters. The obtained values of the optimizable vector can then be used to build the identifiable function from some class of functions with the use of interpolation and approximation methods. We give the results of the carried out numerical experiments.

Non-stationary motion of incompressible fluid with constant density ρ and viscosity υ over a linear horizontal section of a pipeline of the length l and diameter d can be sufficiently adequately described by the following linearized system of hyperbolic partial differential equations [1]:

$$-\frac{\partial p}{\partial x}=\rho[\frac{\partial \omega}{\partial t}+\alpha\lambda\omega],\quad -\frac{\partial p}{\partial t}=c^2\rho\frac{\partial \omega}{\partial x},\quad x\in(0,l),\quad t>t_0. \tag{1}$$

Here the functions $p=p\left(x,t\right),\omega=\omega\left(x,t\right)$ determine the regime of the fluid motion, namely, pressure and fluid velocity at the point of the pipeline $x\in(0,l)$ at the point of time $t>t_0$; c is the speed of propagation of acoustic waves in the fluid; α is the linearization coefficient.

It is well-known [2] that the hydraulic resistance coefficient λ in (1) depends on the regime of the fluid flow: laminar or turbulent, i.e. on Reynolds' number $Re=\omega\, d/\upsilon$. The coefficient λ depends also on the relative quantity of the inner surface roughness of the pipeline section $\varepsilon=k/d$. Here k is the quantity of the absolute roughness characterizing the state of the inner surface of the pipeline.

At present, to determine the value of the hydraulic resistance coefficient in pipelines, there exist a great number of empirical and semi-empirical formulas derived on the basis of experimental investigations. When applying these formulas to the hydraulic computation of specific pipelines, there arise well-known difficulties. That is why it is important to carry out investigations in order to determine the real value of the hydraulic resistance coefficient individually for each pipeline system.

The identification problem considered in the problem belongs to the class of inverse coefficient problems with respect to systems with distributed parameters. At that we take into account that the identifiable function of the hydraulic resistance coefficient depends on the current process state, namely, on the fluid velocity. The initial data for the solution to the inverse problem are known regimes of the pressure and/or fluid velocity observed at different points of the linear section of the pipeline continuously or at discrete points of time.

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Studing of the Exact Inverse Problem in Relative Potential and it is Solution

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In this work the wave equation is analytically solved in the variational form and for the gradient of the functional the analytical expression is found. Also analytical expression for optimal potential in inverse potential is obtained. Problem statement. It is known that the motion of a particle in a central field is described by the equation

$$-\frac{a}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{bR}{r^2}+q(r)R=ER. \tag{1}$$

Here $a>0$ and b are given numbers and $q(r)$ is the energy of interaction. Multiplying this equation by the r^2 and denoting

$$Q(r)=b+q(r)r^2$$

we obtain

$$-a\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+Q(r)R=Er^2R. \tag{2}$$

One of the interesting problems is to obtain analytical solution of Eq.(2) for different and special potentials $Q(r)$, which is not always possible. Also one of the main problems is the solution of inverse problem, finding of the potential $Q(r)$ on the given energy eigenvalues of the Eq.(2), if possible:

$$R(r_0)=z_0,\ R(r_1)=z_1,\ R(r_2)=z_2,...,R(r_n)=z_n, \tag{3}$$

where

$$0<r_0<r_1<...<r_n;\quad n\geq 2.$$

Now we consider the equation (2) on the interval $[r_1,r_n]$. Our aim is to find here the potential $Q(r)$ in the interval $[r_1,r_n]$, such that the solution of the problem (2) $R(r)$ satisfies the condition (3). Here we will assume that the solutions of the task posed in (2),(3) exists. In order to solve the problems

(2),(3) we write it in the variational form: Now we will find the minimum of the functional

$$J(Q) = \sum_{i=1}^{n-1} [R(r_i) - z_i]^2 \to \min, \tag{4}$$

under conditions (2) and

$$R(r_0) = z_0, \quad R(r_n) = z_n. \tag{5}$$

Let the class of controls $Q(r)$ has the form:

$$U = \{Q = Q(r) \in L_2(r_0, r_n) : Q_0 \le Q(r) \le Q_1, \ \forall r \in \overset{\circ}{[r_0, r_n]}\}. \tag{6}$$

Using standard techniques of [1] from (4), (5), (6) we obtain the following theorem.

Theorem 1. *Let $Q^* = Q^*(r)$ be the optimal potential for the problem (2), (4), (5). Then for any $Q = Q(r) \in U$ the relations*

$$Q^*(r)\ \psi^*(r)R^*(r) = \min Q(r)\,\psi^*(r)R^*(r),$$

$$\forall r \in [r_0, r_n], Q_0 \le Q(r) \le Q_1, \tag{7}$$

are true. Here $R^* = R^*(r)$, $\psi^* = \psi^*(r)$ solutions of a problem (2), (3), at $Q = Q(r)$.

Theorem 2. *Functional (4) is differentiable and its gradient is given by the formula*

$$J'(Q) = \psi\, R. \tag{8}$$

Theorem 1 allows us to determine the optimal potential analytically.

Consequence. *Let $Q^* = Q^*(r)$ be the optimal potential for the problem (2), (4), (5). Then*

$$Q^*(r) = \begin{cases} Q_0, & if \ \ \psi^*(r) > 0, \\ Q_1, & if \ \ \psi(r) < 0. \end{cases} \tag{9}$$

In the case $\psi^*(r) = 0$, the potential can be chosen arbitrarily.

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Control of the sequential treatment process of many antibiotics

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It is known that the effectiveness of the antibiotic treatment is reduced eventually. So the antibiotic should be replaced. We consider a mathematic model and control problem of the sequential treatment process of many antibiotics. Suppose due to mutations appear bacteria resistant to the action of this antibiotic. This bacteria class becomes the dominant if the organism is not cure. So the given antibiotic has already non-effective. The treatment continues with using another drug. If we abstain for some time from the treatment, then the sensitivity of bacteria to the initial antibiotic is gradually recovering because the general bacteria population is more viable than mutants.

Let us consider n different antibiotics. The system is described by functions $x_i,\ i = 0, ..., n$. It characterizes the evolution of the bacteria number of i-th class, which are resistant to i-th antibiotic, where zero antibiotic is the absence of any drug. These bacteria are sensitive to all antibiotics. The treatment process is divided by sequential stages. It is possible that there exist stages without any drug. There are the stages with zero antibiotic. The considered system is described by the system of nonlinear differential equations

$$\dot{x}_i = \sum_{j=0}^{n} a_i^j x_j - b_i x_i \sum_{j=0}^{n} x_j - f_i^{u_k}(x_i),\ i = 0, ..., n,\ t \in (t_k, t_{k+1})$$

at the k-th stage of the process, where t_k is the begin of k-th stage of the treatment, and u_k is a number of the used antibiotic.

We suppose the birth of mutants, which are resistant to each antibiotic, from each bacteria class. Therefore the first term at the right side of the equations (1) describes the augmentation of the number of i-th bacteria class by the natural birth rate and the mutations of other bacteria classes. The positive value a_i^j characterizes the birth rate of i-th bacteria class from j-th bacteria class. Besides the environment is bounded. So the real augmentation of bacteria number is decreased. This phenomenon is described by the second terms at the

right side of the equations (1). The positive number b_i characterizes the degree of the influence environment boundedness to the augmentation of number for i-th bacteria class. The third summands at the right side of the equations (1) that is the functions f_i^j describe the influence of j-th antibiotic to i-th bacteria class. It depends from the number of the considered bacteria class.

The parameters of the system satisfy some constraints. At first the natural augmentation of bacteria number is appreciably exceeded its augmentation by mutations. So the general bacteria population is more viable than mutants. The linear terms at the right side of the equations (1) correspond to the model of Malthus. It guarantees the exponential augmentation of bacteria population. The quadratic terms are typical for Verhulst's model. It realizes going to the stationary state of the system. This is the maximum of bacteria number, which can exist at this environment. Therefore the influence of the antibiotic has the degree of the augmentation more than two. So we use the formula $f_i^j(x) = c_i^j x^{\theta_i^j}$, where the parameter $\theta_i^j > 2$ describes the influence of the j-th antibiotic to the i-th bacteria class. The constants c_i^j satisfy the conditions $c_i^0 = 0$, $c_i^i = 0$ for all $i = 0, ..., n$. Other parameters c_i^j are equal to $s_j d_i^j$, where s_j is the concentration of j-th antibiotic, and d_i^j is a positive number.

There is the general bacteria class only at the initial stage of the process, i.e. $x_0(t_0) = x_{00}$, $x_i(t_0) = 0$, $i = 1, ..., n$. The infection of the organism is considered at the initial stage. So the antibiotic is not applied so far, i.e. $u_0 = 0$. Besides we change the drug at the each stage, i.e. $u_k \neq u_{k+1}$. The organism is reputed cured if the total bacteria number at a time T is not greater than maximal admissible value x_*.

We have the problem of choosing the optimal strategy of the treatment. It consists in the selection of the starting time of the stage of the treatment, the numbers of the antibiotics, and its concentration for each stage. We take into consideration that the large value of the antibiotic concentration invokes the intoxication of the organism. We use the experimental data of Scientific Center for Anti-infectious Drugs (Almaty).

Optimization heuristic for identification keys

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The biological keys are used to determine the species among huge biodiversity. There are two contrary conditions arise in the construction of the biological keys: size and accuracy. Usually the point of optimization is the accuracy, which is formalized by the probability of mistake (so each comparison of each two organisms on the basis of each property should be supported by the accuracy) [1,2,3,4,5]. Here we consider a much more simple situation: we have a number of organisms and a number of its properties, where each property of each organism lays within a given interval. We put the problem to separate all the organisms by a decision tree of minimal height. Such a model requires rather simple and natural input data, which can easily be compiled by a field researcher. This approach does not involve any assumptions about the probability of mistake; the statistical variation of given properties within given species is enough.

Algorithm (heuristic for the key construction)

1. Let us fix a set of organisms $O \triangleq \{o_1, \ldots, o_n\}$; a set of properties $P \triangleq \{p_1, \ldots, p_m\}$ and a matrix of intervals $A(m,n) = (\mathbf{a}_{ij})$, where $\forall i \in \overline{1,m}, \forall j \in \overline{1,n}$ $\mathbf{a}_{ij} \triangleq (s_{ij}, e_{ij}) \in \mathbb{R}^2$ is the pair of lower and upper bounds of the property p_i of the organism o_j.
2. Let $\mathbf{C}$ be an initial partition of set O ($\mathbf{C} := \{O\}$). Let the function $Pow(\mathbf{C})$ shows the power of the largest subset from $\mathbf{C}$: $Pow(\mathbf{C}) \triangleq \max_{X \in \mathbf{C}} |X|$.
3. Let $\forall p \in P$, $\forall O' \subseteq O$ $Div(p, O')$ be an arbitrary partition $\{O''_1, \ldots, O''_k\}$, which satisfy: a) $\cup_{i=1}^{k} O'' = O'$; b) $\forall i, j \in \overline{1,k}$ $(i \neq j) \Rightarrow (O''_i \cap O''_j = \varnothing)$; c) the intersection of the intervals of property p for any two organisms of any O''_i is not empty (the organisms cannot be separated by property p); d) the intersection of the intervals of property p for any two organisms of any two different O''_i and O''_j is empty (the organisms can be separated by property p).
4. While $Pow(\mathbf{C}) > 1$ repeat
   (a) for each $O' \in \mathbf{C}$ such that $|O'| > 1$:
      i. let $P' := \{p \in P \mid Pow(Div(p, O')) < |O'|\}$; if $P' = \varnothing$, then announce, that O' cannot be separated by the algorithm and exit;

ii. let $P'' := \operatorname{argmin}_{p \in P'} Pow(Div(p, O'))$ (select the properties with the best separation);

iii. select an arbitrary $p' \in P''$ and substitute O' in $\mathbf{C}$ by $Div(p', O')$.

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Influence of stochastic noise on bifurcations of stationary states in systems of ordinary differential equations

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In this study we deal with different models of processes which include stochastic noise. These models are formally described by stochastic differential equations.

Solution of that equation is stochastic process. For solving that equation, we have to introduce concept of stochastic integral, Ito, or Stratonovich type. Specially, we consider stationary distributions in stochastic differential equations, and influence of noise on bifurcations.

Also, we introduce well known Fokker-Planck equation (forward Kolmogorov equation) whose solutions are stationary distributions of dynamical system. We study mainly models which describe growth of population, that can be found in work of Sara Pasquali "The stochastic logistic equation: stationary solutions and their stability".

Further, we analyze models with phenomen of bifurcation, FitzHugh-Nagumo and El-Nino model. The first is model of excitation of neurons in the cell, whilst the second describes climate changes.

Special attention is dedicated to the emergence of bifurcation phenomena close to bifurcation parameter.

Methods of Linearization for Solving Quasi-variational Inequalities

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Let H be a Hilbert space and $F : H \mapsto H$ Lipschitz continuous and monotone operator. We will consider the following quasi-variational inequality (QVI): find $x_* \in C(x_*)$ for which

$$\langle F(x_*), y - x_* \rangle \geq 0, \quad \forall y \in C(x_*), \tag{1}$$

where $C : H \mapsto 2^H$ is a set valued mapping with non-empty closed convex values $C(x) \subseteq H$ for all x in H.

Computational experience has shown that application of the projection operation is justified if $C(x)$ is a simple set. But, if the admissible set has a complicated structure of the type $C(x) = \{y \in C_0 : g_i(x, y) \leq 0, i = 1, \ldots, m\}$, projection becomes too complex an operation, in which case it is better to approximate the set $C(x)$ by family of simpler sets. It seems natural to take approximating families of the admissible set as the family of polygons for an exterior approximation.

Feasible set of QVI (1) is defined by $X = \{x \in C_0 : g_i(x, x) \leq 0, i = 1, \ldots, m\}$ and we can it approximate by polyhedron

$$\Gamma(x) = \{y \in H : g_i(x, x) + \langle \partial_1 g_i(x, x) + \partial_2 g_i(x, x), y - x \rangle \leq 0, \, i = 1, \ldots m\}.$$

Now, iterative method for solving quasi-variational inequality has the form

$$x_{k+1} = \pi_{\Gamma(x_k)}[x_k - \alpha_k F(x_k)], \; k \geq 0,$$

where initial point $x_0 \in C_0$ is given, sequence $\{\alpha_k\}$ is parameter of method. Appropriate continuous method is

$$x'(t) + x(t) = \pi_{\Gamma(x(t))}[x(t) - \alpha(t)F(x(t))], \; t \geq 0.$$

In this talk we will formulate necessary conditions for convergence of proposed method and will prove convergence to the solution of QVI (1). Next, we will prove that sequence $\{x_k\}$ is exponentially stable and will find rate of convergence. The similar result we will get for continuous method.

Consistent Conjectures Are Nash Optimal Strategies in the Upper Level Game

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Conjectural variations equilibrium (CVE) was introduced quite long ago as another possible solution concept in static games. Consider a group of n producers ($n \geq 3$) of a homogeneous good with the cost functions $f_i(q_i)$, $i = 1, \ldots, n$, where $q_i \geq 0$ is the output by producer i. Consumers' demand is described by a demand function $G = G(p)$, whose argument p is the market clearing price. Active demand D is nonnegative and does not depend upon the price. The equilibrium between the demand and supply for a given price p is guaranteed by the following balance equality: $\sum_{i=1}^{n} q_i = G(p) + D$.

Every producer $i = 1, \ldots, n$, chooses her output volume $q_i \geq 0$ so as to maximize her profit function $\pi_i(p, q_i) = p \cdot q_i - f_i(q_i)$. Assume that the agents suppose that their choice of production volumes may affect the price value p. The latter assumption could be defined by a conjectured dependence of the price p upon the output values q_i. If so, the first order maximum condition to describe the equilibrium has the form:

$$\begin{cases} p = v_i q_i + b_i + a_i q_i, & \text{if } q_i > 0; \\ p \leq b_i, & \text{if } q_i = 0, \end{cases} \tag{1}$$

where $v_i = -\partial p / \partial q_i$ is the first order variation coefficient conjectured by agent i.

D e f i n i t i o n 1. A collection $(p, q_1, \ldots, q_n)$ is called an *exterior equilibrium state* for given influence coefficients $(v_1, \ldots, v_n)$, if the market is balanced, and for each i the maximum conditions (1) are valid.

Under quite moderate assumptions about the model's data one can deduce the following result.

Theorem 1. *For any $D \geq 0$, $v_i \geq 0$, $i = 1, \ldots, n$, there exists uniquely the exterior equilibrium $(p, q_1, \ldots, q_n)$ depending continuously upon the parameters $(D, v_1, \ldots, v_n)$. The equilibirum price $p = p(D, v_1, \ldots, v_n)$ as a function of these parameters is differentiable with respect to D and v_i, $i = 1, \ldots, n$. Moreover,*

$p(D, v_1, \ldots, v_n) > p_0$, *and* $\frac{\partial p}{\partial D} = \frac{1}{\sum_{i=1}^{n} \frac{1}{v_i + a_i} - G'(p)}$. Having the latter formula from Theorem 1 in mind we introduce the

C o n s i s t e n c y C r i t e r i o n. At an exterior equilibrium state $(p, q_1, \ldots, q_n)$, the influence coefficients v_k, $k = 1, \ldots, n$, are referred to as *consistent* if the equalities below hold: $v_k = \frac{1}{\sum_{i=1, i \neq k}^{n} \frac{1}{v_i + a_i} - G'(p)}$, $k = 1, \ldots, n$.

Now we are in a position to define the concept of *interior equilibrium*.

D e f i n i t i o n 2. A collection $(p, q_1, \ldots, q_n, v_1, \ldots, v_n)$ is called an *interior equilibrium state*, if for the considered influence coefficients, the collection $(p, q_1, \ldots, q_n)$ is an exterior equilibrium state, and the consistency criterion is valid for all $k = 1, \ldots, n$.

The interior equilibrium existence result is as follows:

Theorem 2. *In the considered model, there exists an interior equilibrium state.*

Theorem 1 allows us to define the following game $\Gamma = (N, V, \Pi, D)$, which will be called the *upper level game*. Here, D is a (fixed) value of the active demand, $N = \{1, \ldots, n\}$ is the set of the same players as in the above-described model, $V = R_+^n$ represents the set of possible strategies, i.e., vectors of conjectures $v = (v_1, \ldots, v_n) \in R_+^n$ accepted by the players, and finally, $\Pi = \Pi(v) = (\pi_1, \ldots, \pi_n)$ is the collection of the payoff values defined (uniquely by Theorem 1) by the strategy vector v.

Now the main result of this paper is as follows. In general, the Cournot conjectures are not consistent in our single commodity market model. In other words, the Cournot conjectures $v_i = -p'(G)$ usually do not satisfy the (nonlinear) consistency system. However, in the upper level game introduced above, the consistent conjectures, determined by the consistency equations bring about the Nash equilibrium.

Theorem 3. *The consistent influence coefficients v_k, $k = 1, \ldots, n$, determined as the unique (for the equilibrium price p) solution the consistency equations, compose Nash equilibrium in the upper level game.*

Hyperspectral Fourier-holography of microobjects

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In [1,2] we have shown for the first time that Fourier-spectroscopy principles can be applied to obtain hyperspectral holographic images of microobjects in polychromatic light. In [3] we have experimentally verified the suggested theory. We recorded hyperspectral holograms with modified FT-spectrometer scheme in which point source detector was substituted with CCD Fig.1 (a). Microobject was placed into one arm of the spectrometer, producing spread diffraction pattern in the recording plane. Spectral interferogram of the complex amplitude of the diffracted wavefield of the object was registered in each pixel.

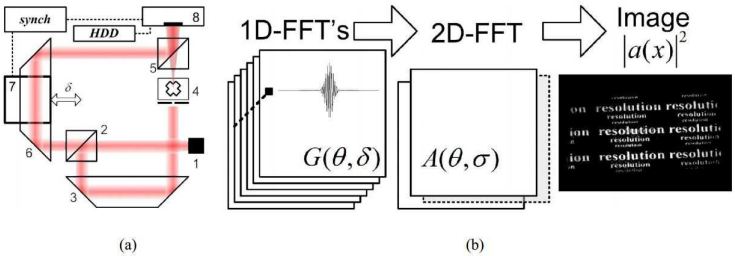

Fig. 1 The principle optical scheme of the hyperspectral holographic Fourier-microscope and image reconstruction computation procedure.

Once 2d interferogram set of $G(\theta,\delta)$ /θ - angular pixel coordinate/ is registered, image construction computation procedure falls into two simple steps Fig.1 (b). Firstly, perform 1d-FFT's over interferograms along δ - dimension for each pixel. Resulting complex data would represent a hyperspectral hologram, consisting of monoholograms, each of which represents complex amplitude of the spatial Fourier-spectrum of the object at a given spectral frequency σ. Thus, performing Fresnel transform over selected monohologram yields a complex amplitude of the object $a(x)$ at selected component σ of the spectrum at the desired distance z. It's important to note that the suggested method enables exact physical color reproduction. To illustrate that we scanned a lotus root, standard microscope stained sample, to synthesize pseudo colored image Fig.2.

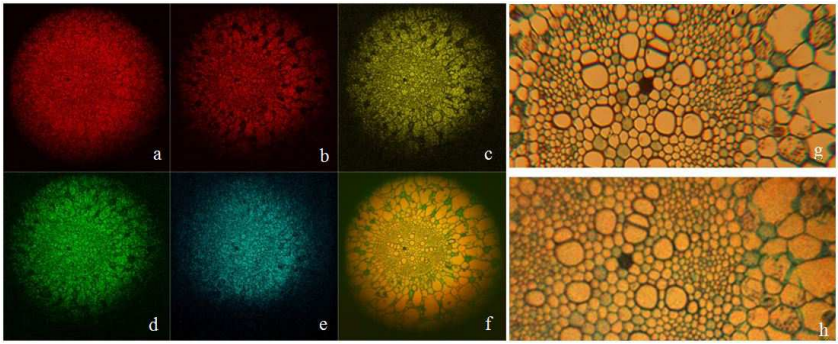

Fig. 2 (a - e) Several monohologram images colored according to CIE 1931 standard, (f) synthesized image from 145 monohologram images. Part of the image from conventional microscope (g) and the same image region obtained on our setup (h).

Summarizing, we'll note that Fourier-spectroscopy, which is by now the most advanced technique of spectral analysis, was one of the first to open the branch of computational optics. Certainly, the method proposed here also bases on technological breakthrough in algorithms for massive computing, optical sensors and other highly advanced digital devices.

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Optimization of the aggregation cancellation fast matrix multiplication scheme

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A technique for the construction of trilinear identities of the type

$$\mathrm{trace}\sum_{l=1}^{L} X_l Y_l Z_l = \sum_{m=1}^{M} f_m(X_1,\ldots,X_L) g_m(Y_1,\ldots,Y_L) h_m(Z_1,\ldots,Z_L)$$

is considered, where $X_1,\ldots,Z_L$ are $n\times n$ matrices, and $f_m(\cdot), g_m(\cdot), h_m(\cdot)$ are linear functions of their entries. For possibly large L and small M, such formula can be converted to fast matrix multiplication $O(N^{log_n(M/L)})$-algorithm (further FMM). For instance, the case of $n=2$, $L=1$ and $M=7$ corresponds to the Strassen algorithm [1], while for arbitrary n and $L=3$ some trilinear decompositions with $M=n^3+12n^2+O(n)$ were found (see [2] and references therein), leading to $O(N^{2.81})$ and $O(N^{2.78})$ FMM algorithms, respectively. Based on paper [3], we present the trilinear identity with $L=3$ and $M=(n+2)^3$, yielding FMM with $O(N^{2.73})$.

With the use of k-dimensional indexing of the arrays, $k\geq 2$, some new trilinear aggregation decompositions are found with $L\geq 3$, which are potentially useful for the construction of even better practical FMM algorithms. The arising discrete optimization problem reduces to finding a special set of L integer permutations of length $3k$.

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Some Euclidean discrete optimization problems and efficient algorithms with performance guarantees for their solutions

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This report is on some discrete optimization problems induced by actual issues, in particular, in approximation theory, computational geometry, and statistics. These problems are also important in data analysis, pattern recognition and in a wide range of applications. For example, they arise in the medical and technical diagnostics, biometrics, electronic intelligence, in the aerospace monitoring, remote sensing of Earth and Ocean, in the criminal science and experimental data processing, and so on. The essence of these problems is to find some informationally important subsets and subsequences in the given noisy data arrays, to partition these data into subsets and subsequences, and to make a decision about objects generating these data. The purpose of the report is to overview new and recent results (obtained in Sobolev Institute of Mathematics) on studying computational complexity of these problems, and on substantiating polynomial algorithms with performance guarantees for solving to these problems.

The main results are of the following sort. Some new (previously unstudied), known (weakly studied), and open problems were investigated. Furthermore, we studied some generalizations and special cases of classical partitioning problems, and also some problems of searching for vectors subsets and subsequences in Euclidean space. NP-hard and polynomial solvable cases of these discrete optimization problems are found. Exact polynomial and pseudo-polynomial algorithms and also efficient approximation algorithms with performance guarantees for these problems are designed.

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An efficient approximation algorithm for a sequence bi-partitioning problem

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We consider following strongly NP-hard [1]

Problem 1-MSSC-S-NF. *Given a sequence* $\mathcal{Y} = (y_1, \ldots, y_N)$ *of vectors from* $\mathbb{R}^q$*, and some positive integer numbers* $T_{\min}$ *and* $T_{\max}$*. Find a subset* $\mathcal{M} = \{n_1, \ldots, n_M\} \subseteq \mathcal{N} = \{1, \ldots, N\}$ *such that*

$$\sum_{j \in \mathcal{M}} \|y_j - \overline{y}(\mathcal{M})\|^2 + \sum_{i \in \mathcal{N} \setminus \mathcal{M}} \|y_i\|^2 \to \min,$$

where $\overline{y}(\mathcal{M}) = \frac{1}{|\mathcal{M}|} \sum_{i \in \mathcal{M}} y_i$*, under constraints*

$$1 \le T_{\min} \le n_m - n_{m-1} \le T_{\max} \le N, \ m = 2, \ldots, M,$$

on the elements of $\mathcal{M}$*.*

In this work we present a 2-approximation efficient algorithm for the problem 1-MSSC-S-NF. This algorithm implements a dynamic programming scheme and runs in time $O(N^2(T_{\max} - T_{\min} + q))$, where $T_{\max} - T_{\min} < N$. Therefore, algorithm is polynomial and it finds the solution in $O(N^2(N + q))$ time in the general case, and in $O(qN^2)$ time in the special case when $T_{\min} = T_{\max}$.

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An exact pseudopolynomial algorithm for a bi-partitioning problem

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In this paper we consider the following strongly NP-hard [1]

Problem. *Given a set* $\mathcal{Y} = \{y_1, \ldots, y_N\}$ *of vectors from* $\mathbb{R}^q$ *and a natural integer number* M*. Find a partition of* $\mathcal{Y}$ *into clusters* $\mathcal{C}$ *and* $\mathcal{Y} \setminus \mathcal{C}$ *such that*

$$\sum_{y \in \mathcal{C}} \|y - \overline{y}(\mathcal{C})\|^2 + \sum_{y \in \mathcal{Y} \setminus \mathcal{C}} \|y\|^2 \to \min,$$

where $\overline{y}(\mathcal{C}) = \frac{1}{|\mathcal{C}|} \sum\limits_{y \in \mathcal{C}} y$ *is the center of cluster* $\mathcal{C}$*, under constrain* $|\mathcal{C}| = M$*.*

This problem has the following conceptual interpretation. A data table (i.e. $\mathcal{Y}$) is given containing the results (i.e. $y_1, \ldots, y_N$) of measuring a q-dimensional tuple (i.e. y) of informative numeric characteristics for some material object. The object can be in one of the two states: active (on) and passive (off). In the passive state the values of all characteristics equal to zero, while in the active state the value of at least one of characteristics differs from zero. It is known that during measurements the object was repeatedly switched from the active state to the passive one. The number (i.e. M) of the active states is given. There are measuring errors in the tabular data; and, moreover, the correspondence between the data and the object state is unknown. It is required to find the disjoint subsets $\mathcal{C}$ and $\mathcal{Y} \setminus \mathcal{C}$ corresponding to the active and passive object states using the criterion of minimum sum of squared distances, and to estimate the values of the object characteristics (i.e. $\overline{y}(\mathcal{C})$) in the active object state using the data from the table.

In [2], a 2-approximation algorithm for the problem is proposed. The running time of the algorithm is $O(qN^2)$. In [3], a polynomial-time approximation scheme with a $O(qN^{2/\varepsilon+1}(9/\varepsilon)^{3/\varepsilon})$-time complexity, where ε is an arbitrary relative error, is substantiated. In [4], a randomized algorithm for the problem is presented. The running time of the algorithm for the fixed failure probability, relative error of the solution and for the certain value of parameter k is

$\mathcal{O}(2^k q(k+N))$. The algorithm has also been proven to be asymptotically exact and to have $\mathcal{O}(qN^2)$-time complexity for the special values of the parameters.

In this work we prove that the problem is solvable in time $\mathcal{O}(q^2 N^{2q})$, which is polynomial in the case of fixed q. We also prove that the general case of this problem does not admit fully polynomial time approximation scheme (FPTAS) unless $\mathcal{P} = \mathcal{NP}$.

Furthermore, we present a pseudopolynomial algorithm which finds an optimal solution in the case of integer values of the components of the vectors in the input set and fixed space dimension. The running time of the algorithm is $\mathcal{O}(qN(2MD+1)^q)$, where D is the maximum absolute coordinate value of the vectors in the input set. It is shown that in the case of fixed space dimension this algorithm is faster then the known polynomial time algorithm if $MD < N^{2-\frac{1}{q}}$.

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Lattice of Cubes, its basic properties and application in combinatorial optimization

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Lattice of Cubes (Cubes Lattice) [1, 2] and its basic properties are discussed. The total number of elements of such lattice (including the empty set) is equal to $3^M + 1$, where M is dimension of the lattice. The investigation of main properties of Cubes Lattice has shown that it is a lattice with supplement and that it is modular and distributive. These properties allow to use previously elaborated effective algorithms [3, 4, 5] for solving on it problems of minimization and maximization of supermodular functions. Some examples of such functions are given and the possibilities of setting and solving a new class of problems on the Cubes Lattices are shown. The following theorems are proved:

Theorem 1. *The total number of subcubes of all dimensions in an m-dimensional cube C^m is 3^m and the number of elements of the corresponding Cubes Lattice K is $3^M + 1$.*

Theorem 2. *The Cubes Lattice K is a lattice with relative supplement.*

Theorem 3. *The Cubes Lattice K is modular and distributive.*

Two algorithms for constructing diagrams of a set of all $C \in K$ (i.e. — Lattice of Cubes), based on this definition, have been worked out.

First algorithm. At a zero level of the Lattice of Cubes diagram there is always only one element — an empty subset ($\varnothing$). At the first level we shall arrange all 2^m sub-cubes containing only one element, i.e. cubes $C^0 = (\omega_1; \omega_2)$, where $\omega_1 = \omega_2$. Then, using the union operation, we shall construct elements of the second level. At this level we shall arrange only cubes C^1, and so on. This process of constructing the Cubes Lattice is similar to the modified algorithm of successive calculations [3, 4] with modifications corresponding to the union operation. In the process of constructing the next levels of the cubes diagram, we construct only a part of elements, as it is done in the algorithm of successive calculations, rather than construct simultaneously all elements of each level. This allows to effectively apply the corresponding rejection rules while solving the optimization problems [3, 4, 5].

Second algorithm. In the beginning the minimal set $\omega_1 = (0, \ldots, 0)$ is fixed and all sets $\omega_2 \supset \omega_1$ are looking through under the order of increase

in binary system of calculation from $\omega_2 = \omega_1$ up to $\omega_2 = I$. During this we construct the corresponding lattice elements and the ribs connecting each constructed element with its neighborhood. Then we add 1 to ω_1 in binary system, and look through all $\omega_2 \supset \omega_1$ again from $\omega_2 = \omega_1$ up to $\omega_2 = I$. This is repeated until ω_1 becomes equal to I. Thus, we obtain all elements of the Cubes Lattice, passed through all 2^m of independent ways, that allows to solve quickly some kinds of problems of discrete optimization. In addition, this algorithm allows to arrange elements of the lattice on a plane in accordance with 2^m basis vectors.

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k-Minimum Hamiltonian Cycles Problem. Complexity and Approximability

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The well known Traveling Salesman Problem (TSP) is the problem of finding a minimum-cost Hamiltonian cycle in weighted complete n-graph, where for each pair i, j of vertices a weight $w_{i,j}$ (also known as 'distance') is assigned.

TSP is NP-hard in strong sense [1] and remains intractable even in the 'metric' case (in which $w(i,j) \leqslant w(i,k) + w(k,j)$ for every $\{i,j,k\}$) and in Euclidean case [2], where vertices are points in finite-dimension Euclidean space and weights are defined by pair-wise distances between these points.

It is known, that general TSP can not be approximated efficiently, unless $P \neq NP$. The metric subclass of the problem belong to Apx thanks to famous Christofides result [3]. On the other hand, it is known [4] that this subclass does not belong PTAS, unless $P \neq NP$. Therefore, getting down of the best approximation ratio for this problem remains a challenge for researchers up to date.

For Euclidean TSP it is known a polynomial time approximation scheme (PTAS) proposed by S.Arora [5]. This result was extended to some closely related problems (e.g. k-TSP, MST).

We consider k-Minimum Hamiltonian Cycles (k-MHC) problem, which can be considered as a natural generalization of initial TSP. For a fixed k it is required to find a partition $\{C_1, \ldots, C_k\}$ of the given weighted complete graph by vertex-disjoint cycles, such that the overall weight of the cycles C_i takes the minimum possible value.

Our results are the following.

Theorem 1. k-MHC problems is NP-hard in strong sense and remains the same even in metric and Euclidean special cases.

Theorem 2. There is 2-approximation polynomial time algorithm for the metric subclass of k-MHC problem.

Theorem 3. There is a PTAS for Euclidean k-MHC problem in the plain finding $(1 + 1/c)OPT$ approximate solution in time $O(n^3(k \log n)^{O(c)})$.

The last result generalizes the famous Arora's result and can be easily extended to some related problems, e.g. to Euclidean k-MHC problem in d-dimensional space within $d > 2$ and some special cases of k-PSP.

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Terminal Control: Linear-Quadratic Case

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In a Hilbert space, the terminal control problem on a fixed time interval $[t_0, t_1]$ with linear dynamics and a quadratic objective function is considered. This study was initiated by the paper [1], wherein a non-local method, converging by functional, for solving linear-quadratic optimal control problem has been proposed.

In our problem formulation, controls are bounded in the norm of $\mathrm{L}_2^r[t_0, t_1]$, and the trajectories are absolutely continuous. We need to find a control $u^*(\cdot) \in \mathrm{U}$ such that the corresponding trajectory $x^*(\cdot)$ will connect a given point x_0 (at $t = t_0$) with the point x_1^* (at $t = t_1$) that minimizes the objective function:

$$\left\{\begin{array}{r} (x_1^*, x^*(\cdot), u^*(\cdot)) \in \operatorname{Argmin}\{\ \langle Sx_1^*, x_1\rangle + \\ + \int_{t_0}^{t_1} (\langle Q_1(t)x^*(t), x(t)\rangle + \langle Q_2(t)u^*(t), u(t)\rangle)dt \ \mid A_1x_1 \le a_1, \\ \frac{d}{dt}x(t) = D(t)x(t) + B(t)u(t), \quad x(t_0) = x_0,\ x(t_1) = x_1^*, \\ x_1 \in \mathrm{R}^n,\ x(\cdot) \in \mathrm{AC}^n[t_0, t_1],\ u(\cdot) \in \mathrm{U}\}\,. \end{array}\right.$$

We consider the problem of terminal control not as an optimization problem, but as a saddle-point problem. Therefore, we do not use the Pontryagin maximum principle, but instead we offer the approach based on reducing the problem to finding saddle points of the Lagrange function:

$$\mathcal{L}(p_1, \psi(\cdot); x_1, x(\cdot), u(\cdot)) = \langle Sx_1^*, x_1\rangle + \langle p_1, A_1x_1 - a_1\rangle$$
$$+ \int_{t_0}^{t_1} (\langle Q_1(t)x^*(t), x(t)\rangle + \langle Q_2(t)u^*(t), u(t)\rangle)dt$$
$$+ \int_{t_0}^{t_1} \langle \psi(t), D(t)x(t) + B(t)u(t) - \frac{d}{dt}x(t)\rangle dt$$

defined for all $(p_1, \psi(\cdot)) \in \mathrm{R}_+^m \times \Psi_2^n[t_0, t_1]$, $(x_1, x(\cdot), u(\cdot)) \in \mathrm{R}^n \times \mathrm{AC}^n[t_0, t_1] \times \mathrm{U}$, where AC^n is a linear manifold of absolutely continuous functions.

To find the solution (saddle point) we suggest a special saddle-point method

of extragradient type:

1) projected half-step

$$\frac{d}{dt}x^k(t) = D(t)x^k(t) + B(t)u^k(t), \quad x^k(t_0) = x_0,$$
$$\bar{p}_1^k = \pi_+(p_1^k + \alpha(A_1x_1^k - a_1)),$$
$$\frac{d}{dt}\psi^k(t) + D^{\mathrm{T}}(t)\psi^k(t) = -Q_1(t)x^k(t), \quad \psi_1^k = Sx_1^k + A_1^{\mathrm{T}}p_1^k,$$
$$\bar{u}^k(t) = \pi_U(u^k(t) - \alpha(Q_2(t)u^k(t) + B^{\mathrm{T}}(t)\psi^k(t)));$$

2) main half-step

$$\frac{d}{dt}\bar{x}^k(t) = D(t)\bar{x}^k(t) + B(t)\bar{u}^k(t), \quad \bar{x}^k(t_0) = x_0,$$
$$p_1^{k+1} = \pi_+(p_1^k + \alpha(A_1\bar{x}_1^k - a_1)),$$
$$\frac{d}{dt}\bar{\psi}^k(t) + D^{\mathrm{T}}(t)\bar{\psi}^k(t) = -Q_1(t)\bar{x}^k(t), \quad \bar{\psi}_1^k = S\bar{x}_1^k + A_1^{\mathrm{T}}\bar{p}_1^k,$$
$$u^{k+1}(t) = \pi_U(u^k(t) - \alpha(Q_2(t)\bar{u}^k(t) + B^{\mathrm{T}}(t)\bar{\psi}^k(t))), \quad k = 0, 1, 2, ...$$

We have proved the weak convergence of the method in controls, and the strong convergence in the other components to the solution.

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Covering algorithm for the simplest polyhedral separability problem

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We consider a particular type of continuous geometric hitting set which reduces to the following simplest polyhedral separability problem for two finite sets of points in general position in $\mathbb{R}^d$.

P r o b l e m . Given subsets A and B of $\mathbb{R}^d$ the problem is to find the smallest cardinality set K of hyperplanes such that for each pair $a \in A$ and $b \in B$ there is a hyperplane from K which separates a and b.

Let $\mathcal{R}$ be some family of subsets in $\mathbb{R}^d$. Given a range space $(\mathbb{R}^d, \mathcal{R})$ continuous geometric hitting set is to find the smallest cardinality set of points whose intersection with each $r \in \mathcal{R}$ is nonempty. Obviously our problem becomes geometric hitting set for finite VC-dimension range space $(\mathbb{R}^{d+1}, \mathcal{R})$ if we take $\mathcal{R} = \{R_{ab}\}_{a \in A,\, b \in B}$ where

$$R_{ab} = \{[x, \alpha] \in \mathbb{R}^{d+1} : (a, x) > \alpha,\ (b, x) < \alpha\}.$$

One can transform this problem in time $O(|A \cup B|^d)$ into the instance of discrete hitting set for range space $(X, \mathcal{R})$ with $|X| = O(|A \cup B|^d)$. Let κ be its optimal value. For this instance Brönnimann-Goodrich $\Omega((|X| + |\mathcal{R}|)\kappa \log |\mathcal{R}|)$-time $O(\log \kappa)$-approximation technique [1] is applicable. In our talk we discuss $O\left(|A \cup B|^{\lceil \frac{d}{2} \rceil + 1} (\log |A \cup B|)^{\lceil (d+1)/2 \rceil}\right)$-time algorithm based on ball covering procedure. It gives $O(\kappa)$-approximate solution for $d = 2$ when convex hulls of subsets forming the minimum cardinality convex monochromatic partition of $C = A \cup B$ are fat objects.

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Mathematical modelling of the condensed moisture transport from the spillways of large hydroelectric power stations (with some optimization of the operation mode of spillways)

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The functioning of spillways of large hydroelectric power stations (HPP) is usually accompanied by formation of large number of air-water mixtures (condensed moisture). Under adverse conditions, the movement of this water-air cloud at working sites of HPP can complicate the completion of the construction works and operation of power equipment. Amount of precipitation from this cloud at the sites of waterworks, can reach 10 m/day (HPP Kapanda, Angola). During winter there is the opportunity to the intensive icing of facilities near spillway, and to the significant fall of solid precipitation (snow, ice) on some areas that are far from spillway (Sayano-Shushenskaya HPP, Boguchanskaya HPP, Russia).

The report presents the results of researches aiming at the mathematical simulation of the phenomenon mentioned above. Developed in 2007-2013 computer model [1-5] comprises the following three parts:
- the semiempirical model of the inkjet spillway,
- the 3D model of the forced convection of the air near HPP,
- the 3D model of the advection and turbulent diffusion of condensed moisture, including ice formation.

The model was originally developed for the theoretical research of condensed moisture migration in the lower pool of the HPP "Kapanda" located in the middle reaches of the river Kwanza (Angola). After finalization this model was applied to the scientific justification of measures to ensure the safe operation of the Sayano-Shushenskaya HPP spillway on the Yenisei River (Russia) in emergency period 2009-2010. Finally, the model was used to assess the impact of the Boguchanskaya HPP spillways (Angara River, Russia) during the initial period of filling of HPP reservoir in winter 2012-2013. The model also was used to select the optimal operation mode of the spillways in this period.

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Measurement of returns to scale in the radial and non-radial DEA models

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Many scientific papers were devoted to the measurement of returns to scale in radial and non-radial Data Envelopment Analysis (DEA) models. The DEA models possess some specific features that give rise to estimation problems: a) multiple reference sets may exist for a production unit under estimation, b) multiple supporting hyperplanes may occur on optimal units on the frontier, c) multiple projections may occur in the space of input and output variables. An interesting approach was proposed for measurement of returns to scale based on using strong complementary slackness conditions (SCSC) in the radial and non-radial DEA models [1, 2]. However, we showed in our previous papers that the SCSC radial and non-radial DEA models are not efficient from computational point of view. In this work, we propose a two-stage general approach in order to measure returns to scale at projections points in the radial and non-radial DEA models. At the first stage, a relative interior point, belonging to the optimal face, is found using a special elaborated method. In [3] it was proved that any relative interior point of a face has the same returns to scale as any other interior point of this face. At the second stage, we propose to determine the returns to scale at the relative interior point found at the first stage. Our computational experiments documented that the proposed approach is reliable and efficient for solving real-life DEAmodels.

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On comparison of different sets of units used for improving the frontier in the DEA models

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Every mathematical model is just an approximation of the real-life processes and phenomena. For this reason some inadequacies may arise in models. In the Data Envelopment Analysis (DEA) scientific literature several methods were proposed to overcome such difficulties. Some authors proposed to use specific production units in the primal space of inputs and outputs as a starting point in order to improve the frontier of the DEA models.

An empirical method was suggested for discovering suspicious units, they called them exterior units [1]. In papers [2, 3] authors described anchor units and proposed to use them to improve the DEA models. In [4] authors gave a strict definition of anchor units and elaborated method for discovering them.

In our previous papers we introduced the notion of terminal units. It was proved that only terminal units can improve the frontier in any possible case. In this paper, the relationship between all sets of units is established. Our theoretical results are confirmed by graphical examples and also verified by computational experiments using real-life data sets.

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Meeting state constraints for optimal control of parabolic systems

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Optimal control of parabolic systems occurs in many application fields. In particular, the problem of optimal control of heat conductivity processes remains relevant.

The problem becomes more difficult with state constraints. If the state functions are considered to be in the set of continuous functions, the Lagrange multipliers associated with these problems are known to be regular Borel measures. Therefore, while setting up the optimality conditions, measures appear on the right-hand side of the associated adjoint state equations.

This causes difficulties in the numerical approximation of the problems. The solution approach to state-constrained optimal control problems through Lagrange multipliers associated with the state constraint leads to technical difficulties.

Moreover, the state constraints are often violated in the case of open-loop control due to inaccurate approximations and due to outside disturbances. It is possible to use feedback control to overcome these difficulties. Feedback control has certain advantages: it is steadier with respect to disturbances. For example, feedback is successfully constructed on the base of bilinear or quadratic control, integral transformations.

In the optimal control theory there is a fruitful approach to consider not only a particular problem but general classes of problems in an abstract mathematical form. For example, optimal control is very often considered in Banach space [1].

In this work the optimal control problem is considered in a more concrete Banach space of measures. The study of measure dynamics has continued since the appearance of the basic works of L. Schwartz, who introduced the notions of the generalized function or distribution and considered differential equations in distributions or measures. These equations are often used for the description of various nature processes, for example, stochastic processes, processes of mathematical physics or quantum mechanics and others.

Measure-valued right hand sides or boundary conditions in partial differential equations have attracted recent interest due to their role in the adjoint equation for optimal control problems with pointwise state constraints [2]. Impulse control and point control are often described by measures. The dynamic

equations of measures are considered as adjoint equations for the optimal control problem in non-reflexive Banach spaces [3]; dynamics of Lyapunov measure is used for the optimal stabilization of nonlinear systems.

In this work the general optimal control problem has been considered in the space of measures; the general solving principles have been established; necessary conditions for the optimality of normed measure dynamics have been deduced; the methods of solving have been developed for specific optimal control problems of distributed parameters systems with a state constraint in the form of an equality; feedback control has been suggested for satisfying the state constraint; optimal control problems of heart conductivity with the state constraints have been considered as an example; feedback has been constructed by using bilinear control and integral transformation; numerical solution has been proposed based on of the method of moments [4].

It can be seen that differential equations in measures are useful for solving optimal control problems. It is possible to solve various specific problems on the bases of the principles stated for a measure dynamics.

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ISS team scheduling problem

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We consider the International Space Station (ISS) scheduling problem. Cosmonauts and dispatchers have to perform given set of tasks $N = \{1, 2, ..., n\}$ during the flight time. Every task is characterized by such parameters like duration l_i, priority p_i, possible time of commencement $T_i = \{t_{i_1}, t_{i_2}, ..., t_{i_{k_i}}\}$ and complexity d_i. The problem is to distribute the tasks to the time planning horizon $H = \{t_1, t_2, ..., t_T\}$ in the most optimal way: it means performing all set of tasks and achieving the uniform load of all team members. The unit of the time planning horizon equals to 5 minutes.

Mathematical problem.

We have:

1. M cosmonauts and dispatcher;
2. set of tasks $N = \{1, 2, ..., n\}$;
3. time planning horizon $H = \{t_1, t_2, ..., t_T\}$ (k days, h hours per day).

Every task is characterized by:

1. priority $p_i = p_{ij}(t_j), i = 1, ..., n, j = 1, ..., T$;
2. duration l_i;
3. complexity d_i.

If $t_j \in T_i$ then $p_i \neq 0$, else $p_i = 0$.
The problem is to maximize the objective function:

$$max \sum_{i=1}^{n} \sum_{j=1}^{P} x_{ij} p_{ij}(t_j), \tag{1}$$

subject to

$$\sum_{i=1}^{n} \sum_{j=1+ah}^{(a+1)h} x_{ij} d_j \leq C, a = 0, ..., k-1. \tag{2}$$

Binary variables x_{ij} have the following meaning: $x_{ij} = 1$ if the task i is performed at the time t_j, and $x_{ij} = 0$ otherwise.

The exact algorithm was proposed to solve this problem. The algorithm is a superposition of Greedy algorithm and Branch-and-bound [1]. Both algorithms are often used to solve Knapsack Problem which variation, Multiply Knapsack Problem with Initial Capacity, is similar to assigned problem [2].

The proposed algorithm consicsts of 3 stages:

1. sorting tasks (greedy);
2. finding Upper Bound;
3. brunching.

This algorithm was programmed on C++ and it's work with different initial data was analized using received program.

Conclusion:

1. algorithm is exact;
2. algorithm is suitable for parallelize;
3. algorithm is able to develop in the event of a complication of the problem;
4. the number of possible solutions depends on the severity of restrictions.

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Optimal schedule for repair a double-track railroad

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We consider a particular case of railway problems, namely, the optimal scheduling of the train operation by a double-track railroad when one of the segments is under repair works. It is necessary for a set of trains available at the stations to determine time-scheduling and destination routing by railways in order to minimize one of the regular objective function [1,2]. Application of the dynamic programming is effective to solve this problem. In this paper we suggest an exact algorithm. A double railway track lies between points A and B with a set of signal posts on it. In Fig.1 they are marked by points. Let call the subpathes between them the segments. Then to take: k – the number of semaphores between points A and B; n – the number of segments, $n = k - 1$; p – transit time of a train on each segment.

All the segments are numbered from left to right. The upper railway gauge is directed to running from the left to the right, and the lower one - to the oncoming traffic. The trains can not move back and make passages to the neighboaring branch line at the signal posts if the segment is not closed for a repair.

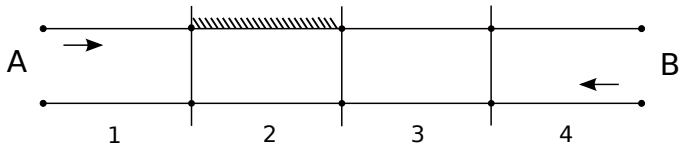

Fig.1

Let us assume that there exist a schedulling π ordering all the trains running to the receiving station to pass each signal post according to the schedule. Because of peculiarities of the railway schedule there exist the time moments when one of the gauge of the segment may be closed. The shaded area in the pictures is the segments where movement is inaccessible. That hereinafter

is referred to as a bottleneck. It is neccessary to create a new scheduling $\pi^{'}$ considering the task with one of the following objective function:

$$\min \sum (C^{'} - C);$$

$$\min \max (C^{'} - C),$$

where $C^{'}$ – value of the aim function for the scheduling $\pi^{'}$; C – value of the aim function for the scheduling π. Moreover, any regular function can be used for this.

Generally, the generation of the states is based only on the information from the previous state. This feature represents wide opportunities for the correlation of the algorithm and it is launching to the multicore architecture of cluster. The idea suggested may be used for the solution of the task of the segment repair planning minimizing the value of any regular objective function and chosing the time period when it is economically profitable. In addition, we suppose that it will not change the complexity of the algorithm dramatically. In this paper we suggested in getting of exact algorithm for the task of the segment repair scheduling.

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Planning algorithm for training cosmonauts in ISS

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We consider the problem of automated scheduling for Internation Space Station (ISS). During planning activities, an important step is the process of preparing cosmonaut to expedition. At this stage, we need to choose which works to teach each of the cosmonauts. Let flight mission comprises a plurality of operations $N = \{1, ..., n\}$ for the set of cosmonauts $M = \{1, ..., m\}$. In this problem, we believe that we know how much time is needed to prepare each of the cosmonaut $j \in M$ on each of the works $i \in N$ and this time equals $p(i, j, q)$, where $q \in \{0, .., , k\}$ is a quality of preparation. An important parameter in this problem is the time C when we must end training and should be ready to fly. Required to arrange work so that the preparation of each participant flight ended in about the same time.

The mathematical formulation

$$\text{minimize} \quad \delta$$

$$\text{subject to} \quad C - \sum_{q=1}^{k} \sum_{i=1}^{n} p(i, j, q)\, x(i, j, q) \le \delta, \quad j = 1, ..., m, \tag{1}$$

$$\sum_{j=1}^{m} \sum_{q=1}^{k} x(i, j, q) = 1, \qquad i = 1, ..., n, \tag{2}$$

$$x(i, j, q) = \{0, 1\}, \tag{3}$$

where $x(i, j, q)$ is a binary variable equalling 1 if work i is given to cosmonaut j and the training goes with quality q, and 0 otherwise.

This problem is $\mathcal{NP}$-hard in the strong sense, thus any dinamic programming approach would result in strictly exponential time bounds [1]. So in this work proposed to use a greedy algorithm with additions.

Lets introduce the additional function δ_i^j [2].

D e f i n i t i o n 1. Functions δ_i^j, $i = 1, ...n$, $j = 1, ...m$ describes the offset from the end of the overall preparation time at j cosmonaut, adding to

his training work i

$$\delta_i^j = \begin{array}{c} \delta_{i-1}^j - p(i,j,k), \text{ if we decide to prepare } j \text{ cosmonaut to } i \text{ work,} \\ \delta_{i-1}^j, \text{ if we don't set } j \text{ work.} \end{array}$$

For all $j = 1, ..., m$ we have the initial conditions $\delta_0^j = C$.

As an objective function for a fixed i choose

$$\text{minimize} \quad \max_j \delta_i^j - \min_j \delta_i^j, \quad j = 1, ..., m,$$

$$\text{subject to} \quad (1), (2), (3),$$

where minimum is selected for the job among all possible permutations. Going all the work we got the smallest variations in the δ for given permutation. The run time of this algorithm is $O(mn)$.

Lets take into account the additional condition $\delta_i^j \geq 0$. If at some stage i this condition is no longer met, the record number of this step in the memory as $b = i$, that is as the value of the present boundary conditions. For b step set to work preparing a lower quality, i.e. such, for which preparations are going faster. Next, the algorithm continues as usual [3].

If either step in what additional condition can not be met under any preparation time, then return to step $i = b - 1$ and put it with the worst quality of training and continue the algorithm. Thus we find the right solution either, or complete inability to scheduling under such initial parameters.

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Application of splines for the analysis of production dependences

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Application of Cobb-Douglas production function [1] and its modifications for the analysis of real macroeconomic processes encounters essential difficulties of methodical character. One of the approaches used for improvement of quality of the forecasting is connected with the use of the differential form of production function and the analysis of its parameters dynamics on the basis of various econometric methods. Development of this approach is connected, in particular, with the researches which were carried out under the supervision of the prof. Yu.P. Ivanilov in the Computer Center of the Russian Academy of Sciences in 1985-1996.

Questions of the analysis of dynamics of elasticity coefficient of dependence of labor productivity from a capital-labor ratio on the basis of a method of a spline-approximation [2] are discussed. The results of spline analysis of statistical data used in paper [1] are given. It is shown that the production sector of the USA from 1899 to 1922 developed in the period unevenly becouse the coefficient of elasticity of Cobb-Douglas production function was variable. The conclusion about expediency of use of capital productivity coefficient with piecewise and constant rates when carrying out practical estimates of economic development is justified.

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Dynamic model of interaction of national income and price level

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In mathematical models of the Keynesian type the equilibrium of the money market and the market for goods and services is determined by the point of intersection of the IS and LM curves, lying in the 'national income - interest rate' plane. In recent years static models of macroeconomic are succeeded by nonlinear dynamic models [1, 2, 3]. In dynamic Keynesian models it is usually supposed that the price level is fixed.

Here we examine several version of discrete nonlinear macroeconomic dynamics model which explores the interaction of national income and price levels. This assumes that the interest rate is fixed, and changes of national income and price level depend on excess demand for goods and services and excess money demand. Examples of computer calculations of trajectories of different variants of models demonstrate opportunity of stable and unstable stationary solutions. It is shown that some version of the model have property of the period doubling bifurcation.

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Multi-objective study of anticancer agents effect on tumor growth by constructing trajectory tubes and Pareto frontier visualization

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The effect of anticancer agents on the cancer tumor growth was studied by using the trajectory tubes constructing and visualization of the multidimensional Pareto frontier (see [1]). This problem has been studied earlier by Italien researchers who performed experiments with nude (young) mice: they implanted the tumor and hailed a mouse by using several anticancer agents [2]. Then, they used the least squares method for identification of the parameters of a mathematical model of the anticancer agents effect on the basis of the experimental results. However, in this case the parameter identification problem cannot be solved correctly since the solution of the identification problem is not stable. For this reason, the Identification Sets Method [3] was used by us for constructing the identification set, that is, the set of parameters, which are in line with the experimental data. Then, the anticancer agents effect model with imprecisely identified parameters was studied.

In the framework of the Identification Sets Method, a visual approach to the identification of the parameters is applied. To be precise, the multidimensional graph of error function is approximated and visualized. By this one can analyze the stability of the solution of the problem of the error function minimization and recognize a stable set of quasi-optimal parameter values. In the study under discussion, the model of the anticancer agents effect on the cancer tumor growth was provided by the system of ordinary differential equations. The parameter identification problem was related mainly to the tumor growth parameters. Thus, a region in the parameter space of the tumor growth that corresponds to the experimental data (the identification set) was constructed by using the Identification Sets Method. Other data were taken directly from the experiment. Since the region in the parameter space was used instead of a unique parameter value, any anticancer agents application strategy resulted in a tube of trajectories.

The study of experimental results in the field of experiments with tumor growth helped to formulate a list of cure strategies used in practice. Decision support in the case of uncertainty in parameter values was based on modification of the Reasonable Goals Method (RGM) that is used in the case of a large

number of multi-objective (multi-attribute) alternatives [1]. RGM is based on visualization of the convex hull of the criterion points related to the decision alternatives. Application of the RGM in the case of alternatives, outputs of which are known not precisely, was started in [4]. Standard Reasonable Goals Method was used for selecting several efficient cure strategies. About 140 strategies of anticancer agents application were considered. Five objectives were taken into account: weight of the tumor after 10 days; weight of the tumor after 10 days; maximal drug concentration in the central part of the body; maximal drug concentration in the peripheral part of the body; week dose.

Then, the tube of trajectories for all selected strategies was constructed. Knowledge concerning these trajectory tubes helped us to approximate their images in the objective space. Visualization of the images of the trajectory tubes helped comparing them, selecting the most preferable strategy and providing the treatment recommendation.

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Parallel Algorithm with Multiple Candidate Solutions

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The complexity of solving global optimization multiextremal problems depends not only on the number of function local optimum, but also on their mutual arrangement. Key concept for the evaluation difficulties solving these problems is the "funnel", first used in chemistry and biology problems. "Funnel" - a set of local minima, such that for each of them there is at least one decreasing sequence, leading to a minimum on the "bottom" of the funnel. The path connecting the two local minima adjacent sequences such that the objective function along it does not exceed a predetermined value. The number of craters with regard to their width may be more appropriate characteristic complexity of the global extremum search. If the function behavior is characterized by a large number of narrow funnel with steep slopes, the search process can never find the global minimum, "stuck" for a long time once the funnel. To overcome this difficulty, it was suggested that if the coating does not begin with a parallelepiped, but their set (family), which correspond to local minima of different craters, it would take a lot less cycles to find the global optimum.

To implement this idea was proposed an algorithm on base of the population approach, which assumes that the set of candidate solutions are supported, evolving in a way that ensures sufficient diversification decisions. The provision of such diversity is achieved by the introduction of measures differences between the solutions which depend on the specific task at hand. The family of candidate solutions can be used in the next step of solving the problem as a starter. In the case of multi-stage calculations, each processor computing cluster at the end of the next stage preserves the totality of its existing parallelepipeds be examined ("trees"), while maintaining the overall record for all processors, and the family record point candidate solutions. Then at the next step the processor-controller distribute these trees among available CPUs.

Experience calculations showed that often effective launch new calculations with the resulting family, but with modified parameters such as the maximum number of family members and the length of a multidimensional cube.

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Damping of oscillations of a string by using multiple point dampers

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The oscillations of a string are described by the equation:

$$y_{tt} = a^2 y_{xx} + g(t,x), \quad 0 <= t <= T, \quad 0 <= x <= l, \quad a = \text{const}.$$

The initial conditions: deviation and velocity – are known:

$$y(0,x) = h_0(x), \quad y_t(0,x) = h_1(x). \tag{1}$$

On the boundary of a string imposes fixing conditions:

$$y(t,0) = y(t,l) = 0.$$

The problem of damping is: to find the control function $g(t,x)$, which allows to get the state of a string from initial state (1) to final state:

$$y(T,x) = 0, \quad y_t(t,x) = 0.$$

This problem also was considered by Lagness [1], Rassel [2], Butkovsky [3]. All these works used the condition of Levinson [4], from which follows theoretical time of damping of oscillations $T = 2l/a$. These approaches show the problem has the solutions, however they are difficult to use in practical applications. In this regard, one considered another approaches including usage of point damper which moves on small part of a string [5].
In this report we consider the damping of oscillations of a string by using multiple point dampers. The function $g(t,x)$ is considered in the form:

$$g(t,x) = \sum_{i=1}^{n} W_i(t)\delta(x - x_i),$$

where x_i – the points on a string where dampers are placed,
$\delta(x - x_i) = \begin{cases} 1, & x = x_i \\ 0, & x \neq x_i \end{cases}$, $W_i(t)$ – control functions.
To find the required control function we use method of gradient descent, to compute gradient we use a method for fast automatic differentiation proposed by Evtushenko [6].

E x a m p l e 1. Let us consider a damping of oscillations of a string with $l = 1$, $a = 1$ by using two dampers, which are placed at $x_1 = 3l/16$, $x_2 = 10l/16$. The initial conditions $h_0(x) = \sin(2\pi x/l)$, $h_1(x) = 0$, $\epsilon = 10^{-4}$. The problem was solved by time $T = 1.2$. Figures 1 and 2 show process of damping oscillations of $y(t, x)$ and the control functions $W_1(t)$ and $W_2(t)$ correspondingly.

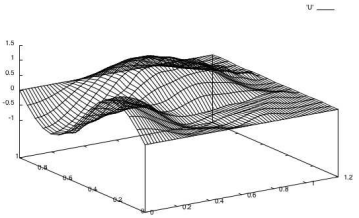

Figure 1: Changing of oscillations by time

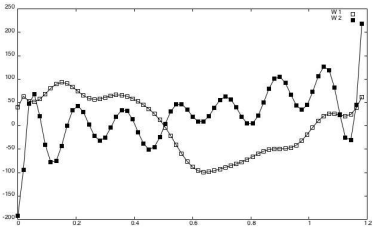

Figure 2: The control functions

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Efficient Organization of Measuring the Mechanical Characteristics of the Human Eye in Clinics on the Basis of a Mathematical Model

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Tonometry is a widely used test for indirect measurements of intraocular pressure in clinics. The implementation includes the mechanical loading of the eyeball by loading devices of different forms and weights and the measuring of the resulting geometrical changes.

The aim of the investigation is to obtain efficient information about the mechanical characteristics and mechanical state of the individual human eye from these data. To solve the above-mentioned inverse problem, a simple model of the eyeball loaded by an external device, which represents the eye shell as the composition of an isotropic, linearly elastic homogenous two-dimensional soft shell (cornea) and an elastic element (sclera), is developed.

The boundary-value problem for the system of four nonlinear differential equations was solved numerically. The condition of boundedness of the tension at the central cornea point leads to an asymptotic relation between the basic parameters, which plays the role of a boundary condition. The solution behavior was investigated depending on the boundary conditions and the system parameters.

It turned out that for certain shapes of the loading device several solutions may exist, one of which only being physically significant. By varying the central tension, among possible shapes of the loaded cornea, the shape for which the distance between the cornea and sclera is minimal was chosen. Using the solution obtained, it is possible to simulate the tonometry test for various tonometer shapes and elastic moduli present in the model and get some useful consequences for clinical testing.

The calculations showed that for loading the cornea by devices of the so-called impression type (convex stamp or thin rod) certain important characteristics depend on the elastic modulus of the sclera alone, whereas for loading by the applanation tonometer (flat stamp) the elastic modulus of the cornea is also important.

The approach developed enables us to find the intraocular pressure and the two elastic constants basing on standard measurements by tonometers of two different types: applanation and impression tonometers.

Separately from the individual elastic properties of the eye (at least scleral

stiffness) the true intraocular pressure cannot be estimated even approximately. Moreover, two elastic parameters are additional characteristics of the eye that can be used in diagnostics. The method proposed is being tested in clinics.

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B-slines Trajectory Planning for Quadrotor Flight

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We present a novel trajectory computation algorithm to construct a quadrotor trajectory through a real-world environment computed by sample-based motion planners. Our approach uses B-splines of order 5 to generate trajectories which are $\mathcal{C}^4$ almost everywhere, except on a few isoled points. The order of B-splines depend on the model of quadrotor. To check the admissible path we use simple quadrotor model [2]

$$\ddot{z} = f_{coll}\cos\phi - g; \ddot{y} = -f_{coll}\sin\phi; \dot{\phi} = \omega_x, \tag{1}$$

where $X = (y, \dot{y}, z, \dot{z}, \phi) \in R^5$ is the system state and $U = (f_{coll}, \omega_x) \in R^2$ denotes system input. Our quadrotor model moves only in the vertical yz-plane with relevant coordinates (y, z and ϕ). So, to construct control we need have continues function $\dot{\omega}_x = f(\partial^4 y^*/\partial^4 t, \partial^4 z^*/\partial^4 t, \dots)$, where $(y^*(t), z^*(t), \phi^*(t))$ is the desired trajectory. It is worth noting that in the full-size model quadrotor will only increase the number of variables.

For example, if $X = (x, \dot{x}, y, \dot{y}, z, \dot{z}, \phi, \beta, \gamma) \in R^9$ is the quadrotor state we have the desired trajectory $(x^*(t), y^*(t), z^*(t), \phi^*(t))$. Two other angles we can reconstruct using equations [3]

$$\beta^* = \arctan\Big(\frac{\ddot{x^*}\cos\phi^* + \ddot{y^*}\sin\phi^*}{\ddot{z^*} + g}\Big),\ \gamma^* = \arcsin\Big(\frac{\ddot{x^*}\sin\phi^* - \ddot{y^*}\cos\phi^*}{\sqrt{\ddot{x^*}^2 + \ddot{y^*}^2 + (\ddot{z^*} + g)^2}}\Big).$$

We don't consider time optimal trajectory for the quadrotor, so there is no reason to use a nonlinear optimization technique. On the other hand it is necessary to propose a criterion for selection of the trajectory candidates.

Suppose given quadrotor flight path, which is obtained using quadrotor GNSS-receiver. First we aproximate each components of this trajectory by B-splines of order 5 using equations

$$P_i(u) = (u^5, ..., u^2\ u\ 1)M_{s+1}(p_i\ p_{i+1}\ \dots\ p_{i+s})^T, \tag{2}$$

where

$$M_{s+1} = [m_{i+1,j+1}]_{i,j=0}^{s} = \frac{1}{s!}C_{s,i}\sum_{m=j}^{s}(s-m)^i(-1)^{m-j}C_{k,m-j}, \tag{3}$$

$C_{i,j} = \dfrac{i!}{j!(i-j)!}$. Here, variable $u \in [0,1]$, points $(p_i\, p_{i+1}\, \ldots\, p_{i+s})^T$ are known. These B-splines curves have jumps of 5 derivatives in the connection points of two adjacent splines. Let $\varepsilon^j = (\varepsilon_0^j, \ldots, \varepsilon_N^j)^T$ are unknown shifts of N points. To minimize the jumps amplitude we need solve the system of algebraic equations $(H + \delta D^j)\varepsilon^j = -H\bar{P}^j$, $j = 0, \ldots, 4$ [1], where $D_j = \mathrm{diag}\{1/\sigma_{1,j}^2, \ldots, 1/\sigma_{N,j}^2\}$, $\sigma_{i,j}$ is the standard error of the GNSS-receiver measurements. We define optimization matrix $H = C_5^T C_5$, where

$$C_5 = \begin{bmatrix} 1 & -5 & 10 & -10 & 5 & -1 & 0 & \ldots \\ 0 & 1 & -5 & 10 & -10 & 5 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

So, we have a new set of points $\hat{p}_i^j = p_i^j + \varepsilon_0^j$, $j = 0, \ldots, 4$, $i = 0, \ldots, N$ and we can construct trajectory candidate as a composition of a parametric form $\hat{P}_i(u_i)$, $u_i \in [0,1]$ and monotonically increasing functions $u_i(t), t \in [t_i, t_{i+1}]$ are known as the motion profile. In simple case $u_i(t) = \frac{t-t_i}{t_{i+1}-t_i}$, $t \in [t_i, t_{i+1}]$.

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On Optimization problem in growing solids

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The present study is devoted to the variational formulation of optimization problem in growing solids. The *imposed* constraints on variable boundary surface are derived by field theory technique and formalism of the calculus of variation.

The general form of action in a variable region of a plane 4-spacetime with the elementary volume of $d^4X = dX^1 dX^2 dX^3 dX^4$ is

$$\Im = \int \mathcal{L}(X^\beta, \varphi^k, \partial_\alpha \varphi^k) d^4X \tag{1},$$

where φ^k is the physical fields array.

The least action principle approves that the actual field is realized in the spacetime in a way that the action of (1) is minimum, i.e. for any admissible variations of physical fields φ^k and non-variable coordinates X^β are $\delta\Im = 0$. Then the classical Euler–Lagrange equations are valid:

$$\mathcal{E}_k(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \varphi^k} - \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\beta \varphi^k)} = 0.$$

The variation of action for *finite* variations spacetime coordinates and physical fields can be represented in the form (see [1])

$$\delta\Im = \int \partial_\beta \left(T^{\beta\cdot}_{\cdot\alpha} \overline{\delta X^\alpha} - \underset{4}{S}{}^{\beta\cdot}_{\cdot k} \overline{\delta\varphi^k} \right) d^4X.$$

where

$$\underset{4}{S}{}^{\beta\cdot}_{\cdot k} = -\frac{\partial \mathcal{L}}{\partial(\partial_\beta \varphi^k)}, \qquad T^{\beta\cdot}_{\cdot\alpha} = \mathcal{L}\delta^\beta_\alpha - (\partial_\alpha \varphi^k) \frac{\partial \mathcal{L}}{\partial(\partial_\beta \varphi^k)}.$$

The first variation of the action can be rewrite after applying the Gauss theorem in form

$$\delta\Im = \oint_\partial \left(T^{\beta\cdot}_{\cdot\alpha} \overline{\delta X^\alpha} - \underset{4}{S}{}^{\beta\cdot}_{\cdot k} \overline{\delta\varphi^k} \right) \mathcal{N}_\beta d^3\tau \tag{2}.$$

Here $\mathcal{N}_\beta$ is the normal 4-vector on boundary surface.

Formula (2) can be further transformed in the case, when the fields and coordinates variations on the variable border region are not independent. Thus variational formulation of the problem includes restrictions in the form of the *imposed* boundary conditions on the variable integrating surface of domain such as:

$$\varphi^k = \Gamma^k(X^\gamma) \quad (\gamma = \overline{1,4}).$$

Then the coordinates variations are coupled with the field variables variations as follows

$$\delta\varphi^k = (\partial_\gamma \Gamma^k)\delta X^\gamma,$$

the formula (2) is rewrite as

$$\delta\Im = \oint\limits_{\partial} \mathcal{N}_\beta \left(T^{\beta\cdot}_{\cdot\alpha} - (\partial_\alpha \Gamma^k) \underset{4}{S}{}^{\beta\cdot}_{\cdot k} \right) \delta X^\alpha d^3\tau,$$

Finally in view of arbitrary and independency of coordinates variations the boundary conditions is obtained:

$$\mathcal{N}_\beta \left(T^{\beta\cdot}_{\cdot\alpha} - (\partial_\alpha \Gamma^k) \underset{4}{S}{}^{\beta\cdot}_{\cdot k} \right) = 0.$$

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New mathematical approach to economic system development management

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The results of the research consisting in mathematic modelling of economic system development held on the basis of physical economy are to be presented in this talk.

This theory claims that all the economic process' parameters should be expressed in a physically measurable quantities. According to this model's basic idea is the division of actual functioning industrial processes that must be described via physically measurable quantities, and its management system implementing different views on economic development management.

Here are some of the model's basics. Suppose there is a set of values $F(t) = \{f(t)\}$ representing commodity flows, and a set of functions $W(t) = \{w(t), w(t) = w(F(t))\}$ representing the industrial processes acting over the given set of commodity flows.

D e f i n i t i o n 1. Let an industrial process w be a function $w(F(t)) \in W(t)$ over the set of values $F(t) = \{f(t)\}$ that causes constant decrease in time of one flows $w(f_i(t)) < f_i(t)$, constant increase of another $W(f_j(t)) > f_j(t)$ and leaves some invariable $w(f_k(t)) = f_k(t)$.

D e f i n i t i o n 2. Let an industrial system be a set of functions $W(t) = \{w(t), w(t) = w(F(t))\}$.

Also the model considers the following important feature that is not considered by the majority of macroeconomic models. The industrial process do not consume all of the needed commodity flows instantly and all of them at one time while functioning. There is a certain delay between flow consumption and another flow output that is different for different flows within a given process.

Importance of those delays is underestimated in present macroeconomic modelling. Well-known Russian economist Mikhail Khazin in his works claims those delays to be one of the key mechanisms that led to the present worldwide economic crisis [4].

Following features distinguish the model from another ones:

- physical economy basis;
- division of economic system on industrial system described via physical economy ideas, and management system that takes any form, from plan economy to free market;
- commodity flows consumption delays in industrial processes.

The model allows to analyze, plan the very within of the economic system – commodity flows. Quite often it is needed to reach some level of consumption which is provided by a certain the set of commodity flows. The model can give a more simple answer to this telling what processes must act, when, how long, how their interaction must be organized. According to that knowledge a suitable management system can be built. It is one of the biggest model's advantages that allows to plan real processes first and then build management system.

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Optimal Regimes Modelling of Global Climate Response to Solar Radiation Management

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Climate change has led to increasingly serious consideration of the potential role of geoengineering as a potential means to avert a climate emergency [1] such as rapid melting of the Greenland and Antarctic ice sheets, or as a stopgap measure to buy time for effective emissions mitigation responses. The general purpose of climate geoengineering proposals is to intervene in the climate system by deliberately modifying the Earths energy balance to reduce potential temperature increases and ultimately stabilize temperatures at levels lower than currently projected. Perhaps the most widely discussed climate geoengineering option is enhancement of planetary albedo (surface reflectivity of suns radiation) using stratospheric sulfate aerosols [2]. The genesis of this approach was a suggestion by Russian climatologist Mikhail Budyko in 1974 [3]. Sulfate aerosols are an important component of the troposphere and stratosphere, and can substantially reduce the incoming solar radiation reaching the Earths system during powerful volcanic eruptions. A study [4] concluded that the amount of sulfur emissions required to compensate for projected warming by 2050 would be between 5-16 TgS/year, increasing to 10-30 TgS/year by the end of the century. In this work, the numerical climate model of intermediate complexity is used. The study is based on a three-dimensional hydrodynamic global climate coupled model, including ocean model with real depths and continents configuration, sea ice evolution model and energy and moisture balance atmosphere model [5, 6]. Optical thickness of scattering stratospheric aerosols in the model depends linearly on their extinction coefficient and aerosol mass per unit. In turn, instantaneous radiative forcing at the top of the atmosphere is proportional to optical thickness with coefficient [4] that is based on measurements performed for the aerosol loading after the Mt. Pinatubo eruption (1991). In the numerical experiments with geoengineering mitigation, global mass of the stratospheric aerosols is modelled via balance between stratospheric sulphur emission and falling per annum. Residence time of stratospheric sulphates is varied from 2 to 3 years [4]. Based on fast mixing of volcanic aerosol clouds in zonal direction, zonal distribution of resulting stratospheric sulphates is assumed to be homogeneous. Aerosol concentration from the year 2010 to 2100 is calculated as a controlling parameter to stabilize mean year surface air temperature. It is shown that by this way it is impossible to achieve the space and seasonal uniform approximation to the existing climate, although it is possible significantly reduce the

greenhouse warming effect. Assumption of a uniform stratospheric aerosol space distribution can stabilize the mean atmosphere global temperature, but climate will be colder at 0.1-0.2 degrees in the low and mid-latitudes and at high latitudes it will be warmer at 0.2-1.2 degrees. Dependence of the mean square deviation (MSD) of atmospheric temperature separately for winter and summer seasons is investigated for optimization purpose. Averaging of results separately in the northern hemisphere, the southern hemisphere and throughout the world for July and January with different values of the aerosol emissions was carried out. On the basis of calculations it is assumed that sulfur emissions from the 2010 year to 2100 year vary linearly. The Pareto frontier is investigated and visualized for two parameters - atmospheric temperature MSD for the winter and summer seasons. The Pareto optimal amount of sulfur emissions would be between 23.5 and 26.5 TgS/year for the northern hemisphere. Similar calculations for the southern hemisphere gave the following best value range: 23.5 - 24.5 TgS/year, and for the whole globe corresponding range: 24.5 - 25.5 TgS/year. Thus, the optimal general emission value would be about 23.5 TgS/year. If geoengineering emissions are stopped after several decades of implementation, their climatic effect is removed within a few decades. In this period, surface air temperature may grow with a rate of several Kelvins per decade. The author was supported by the Russian Foundation for Basic Research (projects no. 14-07-00037, 14-01-00308) and Presidium RAS Basic Research Program (no. 15).

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On optimal choice of the feedback coefficient in the path following problem for a wheeled robot with constrained control resource

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Stabilization of motion of a wheeled robot with constrained control resource along a straight target path is considered. Robot's motion is governed by the nonlinear system (*kinematic,* or *simple-car, model*)

$$\dot{x}_c = v\cos\theta,\ \dot{y}_c = v\sin\theta,\ \dot{\theta} = v\tan\delta/L, \tag{1}$$

where x_c, y_c are coordinates of the target point located in the middle of the rear axle, θ is the orientation angle, L is the wheelbase distance, and δ is the turning angle of the front wheels. The motion of the robot is controlled by turning the front wheels, whereas the forward speed v is assumed to be an arbitrary (perhaps, unknown) function of time $v \equiv v(t)$. Owing to the equation $u = \tan\delta/L$ relating the angle δ with the curvature u of the trajectory described by the target point, we may consider u as the control. Since the turning angle δ is constrained, $|\delta| \le \bar{\delta} < \pi/2$, the control is also constrained: $|u| \le \bar{u}$. Without loss of generality, we may assume that the target path coincides with the x-axis of the coordinate frame. In this case, y_c is the distance from the robot to the target path.

If $|\theta| < \pi/2$, we may take x_c to be the new independent variable instead of time. Denoting it as ξ, introducing the notation $z_1 = y_c$ and $z_2 = \tan\theta$, and replacing differentiation with respect to time in (1) by differentiation with respect to ξ (denoted by the prime), we obtain [1]

$$z_1' = z_2,\ z_2' = (1+z_2^2)^{3/2}u. \tag{2}$$

Let us take control in the form

$$u = -\mathrm{sat}_{\bar{u}}[\sigma(z)/(1+z_2^2)^{3/2}], \tag{3}$$

where $\sigma(z)$ is a linear function with positive coefficients and $\mathrm{sat}_{\bar{u}}(\cdot)$ is the saturation function, which ensures the fulfillment of the control constraint $|u| \le \bar{u}$. It can be seen that, in the region where the control does not reach saturation, the system is linear and exponentially stable. In what follows, we confine our consideration to the special case of function $\sigma(z)$ of the form $\sigma(z) = \lambda^2 z_1 + 2\lambda z_2$, $\lambda > 0$. In this case, the matrix of the linear closed-loop system with the unconstrained control resource has the repeated eigenvalue $-\lambda$, the equilibrium

state $z = 0$ is the stable degenerate node, and any trajectory of the system approaches the origin not intersecting the asymptote $z_2 = -\lambda z_1$.

It is proved in [1] that system (2) closed by feedback (3) is asymptotically stable in the large for any $\bar{u} > 0$ and $\lambda > 0$. Clearly, the rate of convergence to the equilibrium state is greatly affected by the value of λ. Indeed, for "small" values of λ, the saturation in (3) is seldom reached, the system is almost everywhere linear, and the phase portrait is similar to that of the linear system having a stable degenerate node at the origin, with the trajectories exponentially converging to the origin with the rate $-\lambda$. On the other hand, for "large" λ, the convergence rate of the linear system is great, but the control rapidly reaches saturation, so that trajectories consist of many segments where $u = \pm\bar{u}$ (the system is not linear) alternated with short segments with linear behavior, resulting thus in a phase portrait similar to that of the pole-like equilibrium point, which implies frequent switchings of the control between the extreme values, overshooting, and, eventually, slow overall convergence.

Let us pose the following problem: find the interval of λ's for which any trajectory of the closed-loop system intersects the asymptote at most once. By taking values of the feedback coefficient from this interval, we obtain systems whose phase portraits preserve properties of that of the linear system with a stable node. The fastest convergence is achieved for the greatest value of λ in this interval, which is referred to as the optimal feedback coefficient.

Theorem. *The optimal feedback coefficient is given by* $\lambda_{opt} = 3\sqrt{3}\bar{u}/2$.

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Economic Management in terms of nonlinear dynamics

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Economic operates objective regularities of complex systems. We emphasize the following main features of the behavior of complex dynamical systems [1,2] with respect to Economic:

1. Unrecoverable uncertainties of system behavior;

2. The possibility of self-organization in a dynamic system;

3. Strong dependence on the initial conditions, which implies the impossibility of predicting the long period the exact state of the system;

4. Fuzzy relationship between control actions and reactions of the system.

Take a closer look each of these laws.

1. Uncertainty and associated elements of the chaotic behavior is inherent in the evolution of a complex system. Chaotic behavior generates new directions of development. Some of them are deadlock, but the other part realizes new direction of progress. Artificial elimination of uncertainty leads to stagnation and degradation of the system, so it eliminates the emergence of progressive development areas. These postulates are applicable to all complex systems, including the economic and social. From the perspective of economy unrecoverable uncertainty means the impossibility detail and formalization of the production process beyond a certain level. For example, it is impossible to plan exactly time implementation of individual intermediate stages. At the same time, there are attempts to establish an excessive degree of determinism of functioning in the production process. It is impossible to completely abandon the planning and registration in production process, but the level of detail cannot be overly detailed. Complex system reject such influence over granular control.

2. Element of the "protection" of a dynamical system from inadequate management is the self-organization. Economic system is a great example of a self-organizing system. No instructions can significantly affect this self-organizing and self-regulating process. And in this situation it is better control, which supports the principle of " Do not disturb the system to self-organize."

3. The strong dependence on initial conditions (the butterfly effect), which implies the impossibility of predicting for a long period the exact state of the system. Economic corresponds to the meaninglessness of detailed planning for a relatively long time. Analogue of this situation from the standpoint of system dynamics - the senselessness of the long-term forecasts in meteorology, long

quotations of financial instruments, economic condition, etc.

4. The complex system is characterized by fuzzy relationship between control actions and reactions of the system. J. Forrester [3] noted that "the complex systems are peculiar to specific responses - this is the reason of frequent setbacks, and failures of attempts being made to improve the behavior of the system." J. Forrester also notes the set of properties of complex systems: 1) intuitiveness 2) insensitive to changes of many parameters of the system, 3) resistant to administrative innovations, 4) contain points of influence in unexpected places, which are responsible for changing the balance in the system, and etc. The resistance to administrative innovations. Forrester notes that "Complex systems resist most administrative activities. Even in the case where the system is made considerable change its behavior often remains unchanged. The reason lies in the nature of complex systems, counter-intuitive, and their insensitivity to the parameters change."

Unfortunately, modern realities represent numerous examples of ignoring by management basic laws of complex systems. Invent a perpetual motion machine for a long time is prohibited and now desirable to stop violating the laws of nonlinear dynamics in economy management.

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Multiobjective quadratic optimization

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We consider multiobjective optimization problems with all objective functions being quadratic (but not necessarily convex) subject to one quadratic constraint. In this case the Pareto set can be nonconvex as well, but its approximation can be effectively constructed. If all objective functions and the constraint are quadratic forms, the calculations require finding eigenvectors of linear combinations of matrices. If we deal with nonhomogeneous quadratic functions, each iteration is reduced to solution of a scalar nonlinear equation.

There are two reasons to address such problems. The first is design of test examples for Pareto set investigation. We are able to generate tests with arbitrary dimension and arbitrary number of objective functions with known Pareto boundary in nonconvex framework. It gives the opportunity to compare effectiveness of various numerical techniques. The need for such tests is emphasized in the recent survey on multiobjective optimization [1].

The second reason is the extension of trust-region methods of scalar optimization to multiobjective case. In a current point the objective functions are approximated by second-order expressions, while trust region is taken as a ball centered at the point. Points generating the Pareto set of such subproblem are taken as new approximations. Methods of this kind could be competitors to well developed algorithms for Pareto set investigation [1 – 3]

The research is closely related to description of images of sets under nonlinear transformations [4,5]. For instance, the result [4] guarantees the convexity of Pareto set for some two-objective quadratic problems, while "small-ball" theorem from [5] explains probability of convexity in randomly generated quadratic problems.

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On alternative duality and lexicographic correction of right-hand-side vector in improper linear program of the 1st kind

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Consider the linear program

$$\min\{(c,x)\colon Ax=b,\ x\geq 0\}, \tag{1}$$

where $A=(a_{ij})_{m_0\times n_0}$, $c\in\mathbb{R}^{n_0}$ and $b\in\mathbb{R}^{m_0}$ are given, $x\in\mathbb{R}^{n_0}$ is the vector of variables, $(\cdot,\cdot)$ means the scalar product.

Admit program (1) to be improper of the 1st kind [1]. Being infeasible this program has no ordinary solutions. Therefore it must be subject to some adjustment (correction). Such adjustment should be as simple and minimal as possible. It is clear that such properties can be attained only by means of formal mathematical methods.

One of the known approaches to the adjustment above is as follows.

Let

$$A_1x=b_1,\quad A_2x=b_2,\quad \ldots,\quad A_mx=b_m \tag{2}$$

be a splitting of the initial system of equations onto subsystems, which are ranked according to their correction permissibility degree. Let us introduce the sequence of the sets

$$X_1=\operatorname{Arg}\min_{x\geq 0}\|A_1x-b_1\|,\quad X_2=\operatorname{Arg}\min_{x\in X_1}\|A_2x-b_2\|,\quad \ldots\ ,$$

$$X_m=\operatorname{Arg}\min_{x\in X_{m-1}}\|A_mx-b_m\| \tag{3}$$

and approximate the initial improper program by the feasible one

$$\min\{(c,x)\colon x\in X_m\}. \tag{4}$$

Let $\bar{x}$ be the solution of (4) with minimal Euclidean norm. We call it the generalized solution of (1).

Problems (3), (4) are numerically unstable. Fortunately, to find $\bar{x}$ it is not necessary to solve them directly. Instead, we propose to use a new method which is based on the classic Lagrangian symmetrically regularized by A.Tikhonov [2], see also [3].

Namely, set

$$\mathcal{L}^{\alpha}(x,y) = (c,x) - (y,Ax-b) + \frac{\alpha_0}{2}\|x\|^2 - \sum_{s=1}^{m}\frac{\alpha_s}{2}\|y_s\|^2, \tag{5}$$

where the vector of dual variables y is split onto subvectors $y_1, \ldots, y_m$ according to (2), $\alpha_0, \ldots, \alpha_m$ are small positive parameters. Using $\mathcal{L}^{\alpha}(\cdot,\cdot)$ let us write a pair of convex programs

$$\min_{x\geq 0}\max_{y}\mathcal{L}^{\alpha}(x,y) = \min_{x\geq 0}\left((c,x) \;+\; \frac{\alpha_0}{2}\|x\|^2 \;+\; \sum_{s=1}^{m}\frac{1}{2\alpha_s}\|A_s x - b_s\|^2\right),$$

$$\max_{y}\min_{x\geq 0}\mathcal{L}^{\alpha}(x,y) = \max_{y}\left((b,y) - \frac{1}{2\alpha_0}\|(A^T y - c)^+\|^2 - \sum_{s=1}^{m}\frac{\alpha_s}{2}\|y_s\|^2\right).$$

Due to strongly convexity-concavity of (5) these programs are always in the relation of a perfect duality. They give us an interesting example of an alternative symmetric duality scheme for improper linear programs.

In our report we describe the connection between the solutions of the convex programs above. We also present the precise conditions on the parameters of regularization α_s which provide a convergence of x-component of saddle points of the function (5) to the generalized solution $\bar{x}$ of program (1) as these parameters $\alpha_s \to +0$.

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Optimization problems arising in modelling of semi-conductors

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The talk starts with a brief introduction to compound semiconductors, nanostructures and atomistic modelling. Details are given regarding the types of atomic lattices and the properties of the materials that are to be modelled.

Nowdays two main methods for simulating semi-conductor materials are used: quantum mechanics and molecular dynamics. Quantum mechanics is a very accurate approach that gives results close to the experimental data. However due to the huge computational complexity this method is only applicable to relatively small (10^2) groups of atoms which is not sufficient for realistic modelling.

The common way to model material properties is molecular dynamics simulation. This method relies on potential energy function (PEF) that allows to compute the force field of the atomic lattice. Currently there exist many potential energy functions. Most of them have several parameters. The selection of parameters defines properties of the particular modeling material.

The problem of finding PEF parameters ("fitting") can be viewed as an optimization problem. The reference properties of a concrete material are obtained from experiments and/or quantum simulation. The fitting problem consists in finding such values of PEF parameters that provide values of modeling properties most close to the reference ones. Since the analytic representation of modeling proerties are not avialable for most potentials the problem is a classical black-box. In the talk we outline existing methods for resolution of fitting problem and present experimental results for Tersoff potential.

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The p-order sensitivity analysis of degenerate optimization problems

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Consider the following optimization problem

$$\min \varphi(x) \quad \text{subject to} \quad F(x) = 0, \tag{1}$$

where $F : X \to Y$, X, Y – B-spaces, $\varphi : X \to \mathbb{R}$, $F \in \mathcal{C}^{p+1}(X)$, $\varphi \in \mathcal{C}^2(X)$ and at the solution point to (1) x^*

$$\mathrm{ImF'(x^*)} \neq \mathrm{Y}, \tag{2}$$

i.e. $F'(x^*)$ – singular (or F - nonregular at x^*), $F'(x^*) = 0$.

$$\min \varphi(x, \varepsilon) \quad \text{subject to} \quad F(x, \varepsilon) = 0, \tag{3}$$

$\varepsilon \in [0, \varepsilon_0]$, $\varepsilon_0 > 0$.

$$|\varphi(x, \varepsilon) - \varphi(x)| \leq c \cdot \varepsilon \|F(x, \varepsilon) - F(x)\| \leq c \cdot \varepsilon, \tag{4}$$

$\forall_{x \in B_r(x^*) = \{x \in X : \|x - x^*\| \leq r\}, r > 0,\, c > 0.}$

$G = \{x \in X : F(x) = 0\}$, $G_\varepsilon = \{x \in X : F(x, \varepsilon) = 0\}$.

$x^*(\varepsilon)$– a solution to perturbed problem (3). Does it exist a solution $x^*(\varepsilon)$ to (3) and estimation $\|x^*(\varepsilon) - x^*\|$?

Example 1. Consider the problem

$$\int_0^\pi \dot{x}_1^2(t) dt \to \min \tag{5}$$

subject to

$$(\ddot{x}_1(t))^2 - (x_2(t))^2 = 0, \quad x_{1,2}^2(0) = x_{1,2}^2(\pi) = 0,$$
$$x = (x_1, x_2)^T \in W_{2,1}[0, \pi],\ x_1^*(t) = x_2^*(t) = 0.$$

Perturbed problem is

$$\int_0^\pi \left[\dot{x}_1(t) - \varepsilon^2 \cos \frac{t}{\varepsilon} \right] dt \to \min \tag{6}$$

subject to

$$(\ddot{x}_1(t))^2 - (x_2(t))^2 - \varepsilon \cos \frac{t}{\varepsilon} = 0,\ x_{1,2}^2(0) \le c \cdot \varepsilon,\ x_{1,2}^2(\pi) \le c \cdot \varepsilon,\ c > 0.$$

The solutions to (6):

$$x_1^*(\varepsilon, t) = \varepsilon^3 \sin \frac{t}{\varepsilon},\ x_2^*(\varepsilon, t) = \pm \sqrt{\varepsilon \sin \frac{t}{\varepsilon} - \varepsilon^2 \sin^2 \frac{t}{\varepsilon}}$$

$$\varphi(x) = \int_0^\pi \dot{x}_1^2(t) dt, \quad \varphi(x, \varepsilon) = \int_0^\pi \left[\dot{x}_1(t) - \varepsilon^2 \cos \frac{t}{\varepsilon} \right]^2 dt.$$

But

$$x_2^*(\varepsilon, t) \nrightarrow 0 = x_2^*(t),\ \varepsilon \to 0$$

i.e.

$$x^*(\varepsilon, t) \nrightarrow x^*(t),\ \varepsilon \to 0\ !$$

– nonstable.

$$\int_0^\pi \left(\dot{x}_1^2(t) + \dot{x}_2^2(t) \right) dt \to \min \tag{7}$$

subject to

$$(\ddot{x}_1(t))^2 - (x_2(t))^2 = 0,$$

– is stable !

Our work is devoted to these questions.

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Constructing the Set of Effective Vector Estimates and the Problem of Comparing the Approximations

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We consider the first-order numerical method for construction a sequence of approximations of the set of effective vector estimates, which does not require a continuum of the solutions of optimization problems. Proposed the theoretical foundation of the method, theorem on the convergence (in the Hausdorff metric) of the sequence of approximations to a set of effective vector estimates is proved. Proposed the modification of the method in case of a large number of partial effectiveness criteria, based on the principles of decomposition and secondary parameterization. To compare the different approximations of the same set of effective vector estimates, the unary and binary indicators were proposed, and the quality of approximation calculated by integration (by the standard simplex) of maximum of normalized Germeiers convolution. Approximation, from which set of effective vector estimates deviates slightly, are preferable to any other by the both indicators. To calculate the values of indicators by the numerical integration methods, was solved the problem of partition the domain of integration (multi-dimensional standard simplex) by uniform cubic grid. It allows us to consider introduced indicators as workable tools for assessing the quality of approximations.

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Optimal balanced circular packing problem

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We consider the balanced packing problem of a given collection of circles into a containing circle of the minimal radius as a multiextremal nonlinear programming problem. We discuss here an algorithm to find local minima and an algorithm to improve a lower bound of the global minimum of the objective function [1]. The algorithms use non-differentiable optimization methods based on Shor's r-algorithm and Shor's dual bound ψ^*. Numerical test results are given.

Let a collection of circles S_i, their radii r_i and weights w_i, $i = 1, \ldots, m$, be given. We assume that the gravity centre of circle S_i is placed at its centre. We denote a variable centre of circle S_i by (x_i, y_i) and a variable radius of containing circle S by r. Let $\lambda_i = w_i / \sum_{i=1}^{m} w_i$, $i = 1, \ldots, m$, and $r_{low} = \max_{i,j}(r_i + r_j)$ be the evident lower bound on r.

Optimization problem. Pack a collection of circles S_i, $i = 1, \ldots, m$, into circle S of minimum radius, subject to the gravity centre of the collection of circles is located at the centre of the circle S which coinsides with the origin. We formulate the balanced circular packing problem as the following multiextremal nonlinear programming problem:

$$r^* = \min_{x,y,r} r \tag{1}$$

subject to

$$x_i^2 + y_i^2 \le (r - r_i)^2, \quad i = 1, \ldots, m, \tag{2}$$

$$(x_i - x_j)^2 + (y_i - y_j)^2 \ge (r_i + r_j)^2, \quad 1 \le i < j \le m, \tag{3}$$

$$\sum_{i=1}^{m} \lambda_i x_i = 0, \quad \sum_{i=1}^{m} \lambda_i y_i = 0, \tag{4}$$

$$r \ge r_{low}, \tag{5}$$

where $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$.

Using nonsmooth penalty functions we reduce the original problem to an unconstrained nonsmooth minimization problem. To solve the problem (1)–(5) we use the combination of Shor's r-algorithm and multistart to find "good" local extrema for the unconstrained minimization problem.

Shor's dual bound ψ^* [2] gives a low approximation of the objective function in the quadratic extremal problem

$$f^* = (r^*)^2 = \min_{r,x,y} r^2, \tag{6}$$

subject to

$$x_i^2 + y_i^2 - r^2 + 2r_i r - r_i^2 \le 0, \quad i = 1, \dots, m, \tag{7}$$

$$-x_i^2 + 2x_i x_j - x_j^2 - y_i^2 + 2y_i y_j - y_j^2 + (r_i + r_j)^2 \le 0, \quad 1 \le i < j \le m, \tag{8}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i \lambda_j x_i x_j = 0, \quad \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i \lambda_j y_i y_j = 0, \tag{9}$$

$$r^2 - (r_{low} + r_{up})r + r_{low} r_{up} \le 0. \tag{10}$$

where constraints (7) and (8) correspond to constrains (2) and (3), two equations in (9) correspond to equations (4). Quadratic inequality (10) comes from $r_{low} \le r \le r_{up}$, where r_{up} may be found by the solution algorithm for problem (1)–(5). A value of ψ^* can be found with arbitrary accuracy by methods of minimization of non-smooth convex functions [2].

Inequality (10) provides nontrivial, i.e. not equal to $-\infty$, dual bound ψ^*, which is a low bound of f^* in problem (6)-(10). Moreover, $\psi^* \ge (r_{low})^2$. Therefore, the first algorithm allows us to improve the value of r_{up}, while dual bound ψ^* improves the value of r_{low}. Indeed, relation $r^* \ge \sqrt{\psi^*} \ge r_{low}$ comes from $\psi^* \le f^* = (r^*)^2$. It is clear now that if value $\sqrt{\psi^*}$ is greater than r_{low}, then it can be used for improving r_{low}. It means, that bound $\sqrt{\psi^*}$ can be employed to prove that the global minimum can be found for the considered class of problems (1)-(5).

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Automatic stabilization for solving the Lagrangian dual problem using column generation

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In a decomposition approach to solving a Mixed Integer Program (MIP), a "tractable" subproblem is identified; it is defined by a subsystem of constraints that typically exhibit a block-diagonal structure; while the remaining constraints are qualified as "complicating" constraints. The problem is then reformulated in terms of variables that specify if a subproblem solution is to be used to form the solution to the full MIP. Column generation is used standardly to solve the linear program that results for such Dantzig-Wolfe reformulation. This so-called master program has indeed as many variables as subproblem solutions. Hence, to cope with such size, one needs to generate them dynamically in the course of the optimization procedure. Column generation is intimately linked to Lagrangian approaches, since the master is the linear programming equivalent of the Lagrangian dual resulting from relaxing the complicating constraints.

The column generation procedure is nothing but an application of the revised simplex algorithm with a specific procedure for pricing. One iteratively solves a version of the master program restricted to a subset of its variables, collects the dual solution, and check the optimality of the current solution by pricing non-basic columns. However, pricing is not done by enumeration of the exponential number of columns, but by optimizing the reduced-cost value of solutions of the subproblem mixed integer polyhedra. This pricing subproblem is precisely that resulting from the Lagrangian relaxation of the complicating constraints. Hence, its solution allows to define a valid Lagrangian dual bound. Thus, the procedure yields a sequence of price vectors (dual solution to the master program) $\{\pi^t\}_t$, where t denotes the iteration counter, converging towards $\pi^* \in \Pi^*$, where Π^* denotes the set of optimal dual solution to the master LP; and associated Lagrangian bound values, $\{L(\pi^t)\}_t$, converging towards the Lagrangian dual value L^*. Stabilization techniques are devised to accelerate the convergence of the dual sequence $\{\pi^t\}_t$ towards $\pi^* \in \Pi^*$.

The column generation procedure does indeed suffer several drawbacks: *dual*

oscillations, *tailing-off effect*, *primal degeneracy*. Some stabilization techniques have been designed to adress these drawbacks, such as penalty function approaches [1,2] and so-called dual price smoothing techniques [3] whereby the price vector used for column generation is defined as a combination of the current dual solution to the master and previous iterate solutions.

From a practical stand point, a key issue is the parameterization of these stabilization techniques. Indeed, the drawback of each of these approaches is to require delicate parameter tuning that is critical for the efficiency of the stabilization scheme. Our work aim at developing automatic-adjustment of the parameters in the course of the algorithm. For dual price smoothing, our parameter self-adjusting scheme uses the sub-gradient information of the Lagrangian function at the current dual solution.

We tested our automatic stabilization approach on a wide range of problems: Generalized Assignment, Multi-Echelon Lot Sizing, Parallel Machine Scheduling, Multi-Commodity Flow, Capacitated Vehicle Routing, Multi-Activity Shift Scheduling, Bin Packing, Cutting Stock, and Vertex Coloring. For the absolute majority of these problems, our approach with the parameter self-adjusting scheme was able to reproduce results of the dual price smoothing stabilization with the instance-wise tuning of the parameter. This automatic approach would be very useful for generic codes which implement the column generation algorithm for solving Mixed Integer Programs.

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Iterative algorithm to the solution of the discrete periodic stabilization problem with incomplete information

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In the work the discrete periodic optimal stabilization problem with incomplete information is considered and an iterative algorithm for its solution is proposed. The results are illustrated by numerical examples.

Let the motion of an object is described by a discrete periodic system

$$x(i+1) = \Psi(i)\, x(i) + \Gamma(i)\, u(i), \quad x(0) = x_0, \quad i = 1, 2, ..., n, \tag{1}$$

with the observation

$$y(i) = C(i)\, x(i). \tag{2}$$

It is required to find the law of regulator

$$u(i) = K(i)\, y(i) \tag{3}$$

minimizing the functional

$$J = \left\langle \sum_{i=0}^{\infty} \left(x'(i)\, Q(i)\, x(i) + u'(i)\, R(i)\, u(i) \right) \right\rangle, \tag{4}$$

where all matrices have proper dimensions.

Thus, we propose the the analog of algorithm [3] for finding the K_i:

1. The initial data $\Psi(i), \Gamma(i), Q(i), R(i), C(i), K_0(i)$ are introduced .
2. $\tilde{\Psi}(0,p), \tilde{Q}(0,p)$ are calculated by the formula

$$\begin{array}{l} \tilde{\Psi}(0,p) = \tilde{\Psi}(p-1)\tilde{\Psi}(0,p-1), \\ \tilde{\Psi}(p-1) = \Psi(p-1) + \Gamma(p-1)K(p-1)C(p-1), \Psi(0,0) = E, \\ \tilde{Q}(0,p) = \tilde{Q}(0,p-1) + \tilde{\Psi}'(0,p-1)\, \tilde{Q}(p-1)\, \tilde{\Psi}(0,p-1), \\ \tilde{Q}(p-1) = Q(p-1) + C'(p-1)K'p-1)R(p-1)K(p-1)C(p-1), \\ Q(0,0) = 0. \end{array}$$

and $S_0(0)$, $U_0(0)$ are determined by the formula

$$\begin{array}{l} S(0) = \tilde{\Psi}'(0,p)S(0)\tilde{\Psi}(0,p) + \tilde{Q}(0,p), \\ U(0) = \tilde{\Psi}(0,p)U(0)\tilde{\Psi}'(0,p) + P. \end{array}$$

3. $S_{j+1}(0)$, $U_{j+1}(0)$ are defined by the formula

$$S_{j+1}(0) = \tilde{\Psi}'(0,p) S_j(0) \tilde{\Psi}(0,p) + \tilde{Q}(0,p),$$
$$U_{j+1}(0) = \tilde{\Psi}(0,p) U_j(0) \tilde{\Psi}'(0,p) + P.$$

4. $S(i)$, $U(i)$ are calculated by following recurrent formulas

$$-S(i) + (\Psi(i) + \Gamma(i) K(i) C(i))' S(i+1) (\Psi(i) + \Gamma(i) K(i) C(i)) +$$
$$+Q(i) + C'(i) K'(i) R(i) K(i) C(i) = 0,$$
$$S(i+p) = S(i), i = \overline{0, p-1},$$

$$-U(i+1) + (\Psi(i) + \Gamma(i) K(i) C(i)) U(i) (\Psi(i) + \Gamma(i) K(i) C(i))'$$
$$+P(i) = 0, i = \overline{0, p-1},$$

where $P(i) = \begin{cases} P, i = p-1, \\ 0, i \neq p-1. \end{cases}$

5. The feedback matrix chain $K(i)$ is calculated by the formula

$$K_{j+1}(i) = -\left(R(i) + \Gamma'(i) S_j(i+1) \Gamma(i)\right)^{-1} \Gamma'(i) S_j(i+1) \times$$
$$\Psi(i) U^{j-1}(i) C'(i) \left(C(i) U^{j-1}(i) C(i)\right).$$

6. Giving a small real number ε, the condition $\|K_{j+1}(i) - K_j(i)\| \leq \varepsilon$ is checked out. If this condition is not satisfied, then we take $K_j(i) = K_{j+1}(i)$ and go to step 3, otherwise the calculation stops.

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About the structure of optimal trajectories of nonlinear objects of the second order

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A problem of optimal control of the object

$$\dot{x} = y, \quad \dot{y} = u(t) - f(x, y) \tag{1}$$

is considered, where $f(x, y)$ is a function of class $C^k(R^2,\, R^1)$, $k \geq 2$, satisfying the conditions

$$f'_y(x,\, y) > 0, \quad \frac{\partial}{\partial y}\left(y f'_y(x,\, y)\right) \geq 0. \tag{2}$$

Any piecewise continuous function $u(t)$, $t \in [0, T]$, with values in $[q, p]$ is said to be an admissible control. Two autonomous systems, received from the system (1) with $u(t) \equiv p$ ($u(t) \equiv q$) are named p-system (q-system).

The structure of optimal trajectories minimizing the linear combination $T + \alpha E$ ($\alpha \geq 0$) of the time T and the consumed energy E for the transference of the object (1) from an initial state to a final state is studied. According to the principle of maximum of L.S. Pontryagin the optimal control may be composed from only the following continuous pieces: $u(t) \equiv p$, $u(t) \equiv q$ and a singular control [1]

$$u(t) \equiv f\big(x(t), y(t)\big) - y^3(t) f''_{xy}\big(x(t), y(t)\big) \times \left[\frac{\partial}{\partial y}\Big(y^2(t) f'_y\big(x(t), y(t)\big)\Big)\right]^{-1} \tag{3}$$

In the case of the singular control (3) the system (1) has the first integral [2] $y^2 f'_y(x,\, y) = \text{const}$.

The optimality of trajectories composed by certain way from the arcs of the trajectories of p-system, q-system and the extremals defined by the equation

$$y^2 f'_y(x,\, y) = \frac{1}{\alpha}, \quad \alpha > 0, \tag{4}$$

is proved.

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Optimization problem for semilinear elliptic equations with nonlinear control

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Optimization control problems for nonlinear infinite dimensional systems with linear controls are well known for small enough values of the set dimension and the velocity of the increment of the nonlinearity. If these parameters are large enough, then the control-state mapping become non-differentiable, and the adjoint system has weak enough properties. The known methods are non-applicable for this case. We could obtain necessary conditions of optimality by means of extended derivatives. However this technique was used for systems with linear controls only. We will use this idea for the equation with nonlinear control.

Consider homogeneous Dirichlet problem for the equation

$$-\Delta y = f(x, v, y),$$

where x is independent variable on the open bounded n-dimensional set Ω, y is the state function, v is the control. It is determined on a convex closed bounded set U of the space $L_r(\Omega)$. Suppose f is Caratheodory function, which satisfies the standard conditions of coercivity, monotony, and boundedness of the increment degree. Then for all function $v \in L_r(\Omega)$ this equation has a unique solution $y = y[v]$ from the space $Y = H_0^1(\Omega) \cap L_s(\Omega)$, besides the mapping $v \to y[v]$ is continuous, where s is the velocity of the increment of the function f with respect to the third argument.

We have the minimization problem for the functional

$$I(v) = \int\limits_{\Omega} F\big(x, v, y[v]\big)dx$$

on the set U, where F is Caratheodory function, which is convex with respect to its second argument and satisfies a condition of the boundedness of the increment degree with respect to its second and third arguments. Using standard optimization methods, we prove the solvability of this problem.

Suppose the function f has the derivatives with respect to the second and third arguments, which satisfy a condition of the boundedness of the increment degree. Using Implicit Function Theorem, we prove the following result.

Lemma 1. *The mapping $v \to y[v]$ is Frechet differentiable for small enough values of n and s.*

Using known optimization methods, we obtain necessary conditions of the optimality.

Theorem 1. *Under the conditions of Lemma* 1 *the optimal control satisfies the variational inequality*

$$\int\limits_{\Omega} \frac{\partial H}{\partial u}(v-u)dx \geq 0 \; \forall v \in U,$$

where $H(x,v,y,p) = f(x,v,y)p - F(x,v,y)$*, and* p *is the solution of the homogeneous Dirichlet problem for the adjoint equation*

$$-\Delta p = -\frac{\partial H}{\partial y}.$$

Lemma 2. *The mapping* $v \to y[v]$ *is not Gateaux differentiable for large enough values of* n *and* s*.*

This result and weak enough properties of the adjoint equation put obstacles in the way of using of the standard optimization methods. The extended differentiability of the control-state mapping was proved before for systems that are linear with respect to the control. However we prove the following result.

Lemma 3. *The mapping* $v \to y[v]$ *is extended differentiable.*

Using this property we obtain necessary optimality conditions for the general case.

Theorem 3. *The assertions of Theorem* 1 *are true for all value of* n *and* s*.*

Thus the extended differentiation theory is applicable for systems with nonlinear control. We can use it for obtaining necessary conditions of optimality if the standard optimization methods are not applicable. We could analyze other optimization control problems, for example, optimization problems for systems described by nonlinear parabolic equations with nonlinear control.

Different degrees of nonlinearity for inverse problem for parabolic equation

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Problem definition and solution.

We consider a problem for nonlinear parabolic equation in one dimension. Mathematical statement looks as following:

$$\partial_t u(t,x) = \partial_x^2 u(t,x) - u(t,x)|u(t,x)|^p + f(t,x), \quad 0 < t < T, \quad 0 < x < 1, \tag{1}$$

$$u(0,x) = \varphi(x), \quad 0 < x < 1, \tag{2}$$

$$\partial_x u(t,0) = b(t), \quad 0 < t < T, \tag{3}$$

$$\partial_x u(t,1) = y(t), \quad 0 < t < T. \tag{4}$$

Function $y(t)$ is unknown and must be determined. For that purposes we use an additional information $u(t,0) = a(t)$. We transform this problem to optimization problem which requires a minimization of a functional

$$I(y) = \int_0^T \big(u(t,0;y) - a(t)\big)^2 dt \to \min$$

If the functional gets its minimum value then $u(0,t)$ is the most close to $a(t)$ and additional information is fulfilled. The most commonly used method for solving such problems is gradient method. For that, we construct a sequence

$$y_{n+1}(t) = y_n(t) - \alpha_n I'\big(y_n(t)\big), \tag{5}$$

where $\alpha_n > 0$.

Here the value of $I'\big(y_n(t)\big)$ is a Gateaux derivative of the functional.

Proofs for linear and nonlinear case of this statement you can see in [1], [2].

Direct problem (1) – (4) can be approximated as:

$$\frac{1}{h^2}u_{i-1}^{j+1} - \Big(\frac{2}{h^2} + \frac{1}{\tau}\Big)u_i^{j+1} + \frac{1}{h^2}u_{i+1}^{j+1} = -\Big(f_i^j + \frac{1}{\tau}u_i^j - \big(u_i^j\big)^3\Big),$$
$$j = 0, 1, \ldots, M-1, \quad i = 1, \ldots, N-1, \tag{6}$$

$$u_i^0 = \varphi_i, \quad i = 0, 1, \ldots, N, \tag{7}$$

$$u_0^j = u_1^j - b_j h, \quad j = M, M-1, \ldots, 1, 0, \tag{8}$$

$$u_N^j = u_{N-1}^j + y_j h, \quad j = M, M-1, \ldots, 1, 0. \tag{9}$$

Adjoint problem required to determine Gateaux derivative can be approximated as:

$$\frac{1}{h^2}\psi_{i-1}^j - \Big(\frac{2}{h^2} + \frac{1}{\tau}\Big)\psi_i^j + \frac{1}{h^2}\psi_{i+1}^j = -\frac{1}{\tau}\psi_i^{j+1},$$
$$j = M-1, M-2, \ldots, 1, 0, \quad i = 1, \ldots, N-1, \tag{10}$$

$$\psi_i^M = 0, \quad i = 0, 1, \ldots, N, \tag{11}$$

$$\psi_N^j = \psi_{N-1}^j + 0, \quad j = M, M-1, \ldots, 1, 0, \tag{12}$$

$$\psi_0^j = \psi_1^j + 2h\big(u_0^j - a_j\big), \quad j = M, M-1, \ldots, 1, 0. \tag{13}$$

Performing experiments. We use the developed numerical algorithm to solve the problem and compare different degrees of nonlinearity (different values of p). First, we understand that the higher degrees of nonlinearity will affect negatively. Numerical algorithm will have bigger errors and that's why the obtained solution can be not so accurate. But on the other side, the functional properties of the equation are better for the higher values of p.

1. High degrees of nonlinearity of equation (1) causes bigger errors after applying numerical algorithms. To compensate that error we need to split time interval on larger number of steps. Accumulated errors cause the maximal discrepancy between numerical solution and exact values of function $y(t)$ for initial time moment, near $t = 0$.

2. Choosing parameters ε and α is a challenge. If we take too small values of α, algorithm converges very slowly, on the other hand, if we take too large values, algorithm diverges. Because of unavoidable errors in algorithm the further decreasing of ε has no meaning and leads to increasing of number of iterations only.

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Solution of some Parametric Inverse Problem of Atmospheric Optics by Monte Carlo Methods

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Problem definition and solution.

In the paper parametric inverse problems of atmospheric optics are considered. To solve these problems we applied algorithm "the method of dependent tests for transport theory problems" of Monte Carlo methods. The problems reduced to linear system of equations for parameters and solved by optimizations methods. The numerical solution of the albedo P_a of the ground surface. The approximation error is no more than 5-10 percent, which is quite satisfactory for Monte Carlo methods. Using the methods above $\sigma_c(\vec{r},\lambda)$ also can be estimated (if $\sigma_c(\vec{r},\lambda)$ is an unknown parameter). Having estimations for $(\sigma_a(\vec{r},\lambda),\ \sigma_m(\vec{r},\lambda),\ \sigma_c(\vec{r},\lambda))$ we can estimate full section by formula

$$\sigma(\vec{r},\lambda) = \sigma_a(\vec{r},\lambda) + \sigma_m(\vec{r},\lambda) + \sigma_c(\vec{r},\lambda).$$

Here $\sigma_c(\vec{r},\lambda)$ is a absorption section, $\sigma_a(\vec{r},\lambda)$ is a section of aerosol scattering with indicatrix $g_a(\mu,\vec{r},\lambda)$, $\sigma_m(\vec{r},\lambda)$ is a section of molecular scattering with indicatrix $g_m(\mu,\vec{r},\lambda)$, $\sigma(\vec{r},\lambda) = \sigma_a(\vec{r},\lambda) + \sigma_m(\vec{r},\lambda) + \sigma_c(\vec{r},\lambda)$ is full section, $g(\mu,\vec{r},\lambda) = \frac{g_a(\mu,\vec{r},\lambda)\sigma_a(\vec{r},\lambda)+g_m(\mu,\vec{r},\lambda)\sigma_m(\vec{r},\lambda)}{|\vec{r}_n-\vec{r}_k|\sigma(\vec{r},\lambda)}$ is full indicatrix, $\tau(\vec{r}_n,\vec{r}_k,\lambda) = \int \sigma(\vec{r}_n+\vec{\omega}_k l,\lambda)dl$ is called "the optical depth" where $(\vec{\omega}_k)l = (\frac{(\vec{r}_k-\vec{r}_n)}{|\vec{r}_k-\vec{r}_n|})l$ is called "the optical length from $\vec{r}_n$ to $\vec{r}_k$", $\vec{\omega}_k$ is unit length vector.

See [1] and [2].

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Bergsonian welfare functions and the integrability problem in demand analysis

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Consider a group of m goods. Denote a consumption bundle of these goods by $X = (X_1, \dots, X_m)$ and their prices by $P = (P_1, P_2, \dots, P_m)$. We say that the inverse demand functions $P(X) = (P_1(X), \dots, P_m(X))$ is rationalizable in functional class Φ of continuous concave positively homogenous and positive in R^m_+ functions if there exists a utility function $F(X) \in \Phi$ such that

$$X \in Arg\max\{F(Y) | \langle P(X), Y\rangle \le \langle P(X), X\rangle, Y \ge 0\}$$

In smooth case inverse demand functions $P(X) \in C^1(R^m_+)$ should satisfy Frobenius integrability conditions $\omega \wedge d\omega = 0$, where $\omega = \sum\limits_{j=1}^{m} P_j(X)dX_j$.

Let′s consider the economic sense of the integrability condition from the view point of neoclassical model of consumer behavior. We assume that M social groups are selected in a society according to their stereotypes of consumer behavior. The stereotype of consumer behavior of the α group describe by the maximizing utility function $u_\alpha(X)$ under the budget constraint $\langle P, X\rangle \le \phi_\alpha(P), X \in R^m_+$ where $\phi_\alpha(P)$ is a share of income of the α group in the total consumption fund. Assume that $\phi_\alpha(P)$ positively homogenous function and $u_\alpha(X)$ belong to the class U_m functions $F(X)$ on R^m_+ such that 1) $F(X) > 0$ for any $X > 0$ 2) $F(X) \in C(R^m_+) \cap C^2(\mathrm{int}R^m_+)$ 3) $F(\lambda X) = \lambda F(X)$ for any $\lambda > 0, X \ge 0$ 4) $F(X) > F(\tilde{X})$ for any $X > \tilde{X} \ge 0$ 5) $\sum\limits_{i,j=1}^{m} \frac{\partial^2 F(x)}{\partial X_i \partial X_j}\nu_i\nu_j < 0$ for any $\nu = (\nu_1, \nu_2, \dots, \nu_m) \ne 0$ such that $\sum\limits_{j=1}^{m} \frac{\partial F(x)}{\partial X_j}\nu_j = 0$. Let $q_\alpha(P) = \inf\left\{ \frac{\langle P,X\rangle}{u_\alpha(X)} \middle| u_\alpha(X) > 0, X \in R^m_+ \right\}$ be price index from the view of group α, $q(P) = (q(P_1), q(P_2), \dots, q(P_m))$.

Let $X^\alpha(P) = (X^\alpha_1(P), X^\alpha_2(P), \dots, X^\alpha_m(P)) \in Arg\max\{u_\alpha(X) | |\langle P, X\rangle \le \phi_\alpha(P), X \in R^m_+\}$ be normalized demand function of group α. Then $X^\alpha(P) = \frac{\phi_\alpha(P)}{q_\alpha(P)} \frac{\partial q_\alpha(P)}{\partial P_j}$. Let′s denote the total demand $X(P) = \sum\limits_{\alpha=1}^{M} X^\alpha(P)$ and $\varsigma = \sum\limits_{j=1}^{m} X_j(P)dP_j$.

Proposition 1. *If* $\varsigma = F(X(P))dQ(P)$, *where* $F(X) \in U_m, Q(P) \in U_m$ *and* $Q(P) = \inf\left\{ \frac{\langle P,X\rangle}{F(X)} \middle| F(X) > 0, X \in R^m_+ \right\}$ *then there exits function* $\Phi(Q) \in U_m$

such that $Q(P) = \Phi(q(P))$ *and* $\phi_\alpha(P) = \frac{\partial_\alpha(P)}{\Phi(q(P))} \frac{\partial \Phi(q)}{\partial q_\alpha}$.

Then Bergsonian function of social welfare is

$$W(u_1, \ldots, u_M) = \inf \left\{ \left. \frac{\sum\limits_{\alpha=1}^{M} q_\alpha u_\alpha}{\Phi(q)} \right| \Phi(q) > 0, q \in R_+^M \right\}.$$

Proposition 2. *If* $\varsigma = F(X(P))dQ(P)$, *where* $F(X) \in U_m, Q(P) \in U_m$ *and* $Q(P) = \inf \left\{ \left. \frac{\langle P, X \rangle}{F(X)} \right| F(X) > 0, X \in R_+^m \right\}$ *then* $F(X)$ *is the optimal value of the functional in the problem*

$$Max \left\{ W(u_1(X^1), \ldots, u_M(X^M)) \Big| \sum_{\alpha=1}^{M} X^\alpha = X, X^\alpha \geq 0, \alpha = 1, 2, \ldots, M \right\}$$

if $X = X(P)$ *for some* $P \in R_+^m$ *(and not less in another case),*

$$X_i^\alpha(P) = \frac{1}{\Phi(q(P))} \frac{\partial \Phi(q)}{\partial q_\alpha}\Big|_{q=q(P)} \frac{\partial q(P)}{\partial P_i},$$

$$u_\alpha(X^\alpha(P)) = \frac{1}{\Phi(q(P))} \frac{\partial \Phi(q)}{\partial q_\alpha}\Big|_{q=q(P)},$$

$$q_\alpha(P) = \Phi(q(P)) \left. \frac{\partial W(u)}{\partial u_\alpha} \right|_{u_\beta = u_\beta(X^\beta(P))(\beta=1,\ldots,M)}.$$

Lets denote $\tilde{u}_\alpha(q) = \frac{1}{\Phi(Q)} \frac{\partial \Phi(Q)}{\partial q_\alpha}$. In condition of proposition 2 we have $u_\alpha(X^\alpha(P)) = \tilde{u}_\alpha(X^\alpha(P))$ and Frobenius integrability conditions $\varsigma \wedge d\varsigma$ (or $\omega \wedge d\omega$) for demand function (inverse demand function) are equivalent to integrability conditions for differential form $\gamma = \sum\limits_{\alpha=1}^{M} \tilde{u}_\alpha(q) dq_\alpha$ for income distribution.

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Approximate local elimination algorithms for sparse discrete optimization problems

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A lot of practical discrete optimization (DO) problems are in the complexity class NP-hard and contain a huge number of variables and/or constraints [1]. So far, there is no solver that can solve such problems in polynomial time. Fortunately, using DO models and algorithms one can solve various applied problems that have a special structure. The matrices of constraints for large-scale problems, as usual, are sparse and the nonzero elements of them often fall into a limited number of blocks. The block form of many DO problems is usually caused by the weak connectedness of subsystems of real-world systems [1]. In this paper integer linear problem with a quasiblock structure is explored.

One of the promising ways to exploit sparsity in the constraint matrix of DO problems is using local elimination algorithms (LEA) [1]. LEA divides problem into separate subproblems and calculates information that characterizes each of them. The local elimination procedure consists of two parts: the forward part eliminates elements, computes and stores local solutions, and computes the value of the objective function; the backward part finds the global solution of the whole problem using the tables of local solutions; the global solution gives the optimal value of the objective function found while performing the forward part of the procedure [2]. LEA is described in detail in [2]. The main drawback of LEA appears from the large size of the separator, which separates the various blocks because a volume of enumeration is exponential on the size of the separator. Relaxations (creating evaluative problems) are often used to reduce volume of enumeration (e.g., linear relaxation and knapsack relaxation).

Using LES (Local Elimination Solver – a framework for solving a class of binary integer linear programming problems and its variants with the help of LEA) we implemented the algorithm with Oracle finding shared variables by solving relaxed problem. Each subproblem is replaced by one-constraint (surrogate) problem, where the constraint is the sum of constraints of the original subproblem. Then problem with new subproblems is solved and values of shared variables are found. We also check if the solution fits constraints of the original problem. After that all shared variables are eliminated from the original problem. So now a range of separate problems is solved.

This work presents computational comparisons of the LEA and combination of LEA and Oracle algorithm. All experimental results were obtained on a

machine with Intel Core 2 Duo processor 2.2 GHz, 3 GB main memory and operating system Linux, version 3.8.0-19-generic. The results can be found in Table 1, in which n denotes the number of constraints, m is the number of variables, k is the number of blocks and s is the number of shared variables in each block, except the first and the last blocks. The next three columns are the difference in values of shared variables, the accuracy of combination of LEA and Oracle algorithm and how much faster it works.

#	n x m	k	s	Difference	*Accuracy, %*	*Times faster*
1	100x300	5	10	*1*	*99,74*	*7,53*
2	100x600	10	8	*4*	*99,50*	*3,42*
3	100x600	10	10	*3*	*99,55*	*3,52*
4	150x200	10	8	*6*	*97,54*	*2,31*
5	150x300	25	8	*16*	*96,71*	*1,27*
6	200x500	10	8	*3*	*99,46*	*6,02*
7	200x500	10	10	*7*	*99,26*	*5,51*
8	200x500	20	6	*8*	*99,31*	*2,04*

Table 1

The computational experiment shows that the help of Oracle algorithm can find only approximate solutions, but it makes significant impact on running time. It seems promising to continue this line of research by studying approximate algorithms that can solve DO problems in polynomial time.

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Algorithms of optimal stabilization of dynamical control systems

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Problem of optimal stabilization to respect to all phase variables and to a part of phase variables for nonlinear dynamical systems is considered in [1] – [5] and in other works. In present report we consider the construction of optimal stabilization algorithms for multiply connected dynamical control systems. Indicated systems are described by multiply connected systems of ordinary differential equations. The basic algorithm of the solving of optimal stabilization problem concerning all phase variables consists of the following stages:

1) to carry out stabilization at the level of the interconnected subsystems, i.e. to find local controls;

2) to present optimal Lyapunov function in the form of a linear combination of functions of Lyapunov for subsystems;

3) to find Krasovsky–Bellman function;

4) to make algebraic system taking into account that on the optimal solution Krasovsky–Bellman function vanishes;

5) to find optimal control for initial multiply connected system from the algebraic system;

6) to define functional in relation to which control is optimal.

By the aid of basic algorithm the optimal stabilization algorithms for various types of multiply connected dynamic systems are offered. Modifications of dynamical models of manipulation robots [6] are considered and for them the corresponding algorithms of stabilization of motion with application of multilevel stabilization [7] are constructed. The method of Lyapunov functions [8] is essentially used for research.

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Triangulations of polyhedral cones and their f-vectors

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Consider a polyhedral cone K, i.e. the set of all solutions to a homogenous system of linear inequalities over the field of rational numbers.

$$\sum_{j=1}^{d} a_{ij}x_j \geq 0 \qquad (i = 1, \ldots, m) \tag{1}$$

The set $K_I = \{x = (x_1, \ldots, x_d) \in K : \sum_{j=1}^{d} a_{ij}x_j = 0,\ i \in I\}$, where $I \subseteq \{1, \ldots, m\}$, is called I-face of K. If $K_I \neq K$ then K_I is called the proper face of K. The set of all proper faces is called the boundary of K. It is known that the set of all faces of K ordered by inclusion is the lattice with minimal face K_0, the intersection of all proper faces of K.

Let B be $(d \times n)$-matrix with columns b_j $(j = 1, \ldots, n)$. Denote by $B^{\angle} = \left\{\sum_{j=1}^{n} b_j y_j :\ y_j \geq 0\right\}$ the set of all nonnegative linear combinations of its columns and suppose that $B^{\angle}$ is coincide with the set of solutions to system (1). Notions and results using here can be found in the literature cited.

The *triangulation of K with knots from B* is the set $T(B) = \{S_1, \ldots, S_t\}$ such that S_τ $(\tau = 1, \ldots, t)$ satisfy the following conditions:

1) $S_\tau \subseteq \{1, \ldots, n\}$, 2) $|S_\tau| = r = \operatorname{rank} B(S_\tau)$, 3) $B^{\angle} = \bigcup\limits_{\tau=1}^{t} B^{\angle}(S_\tau)$,

4) $B^{\angle}(S_\tau) \cap B^{\angle}(S_\sigma) = B^{\angle}(S_\tau \cap S_\sigma)$.

The set $\Delta(T(B)) = \bigcup_{\tau=1}^{t} \Gamma(S_\tau)$ is an example of geometric realization of d-dimensional homogeneous simplicial complex. For $k = 0, \ldots, d$ we denote by $\Delta_k = \bigcup_{\tau=1}^{t} \Gamma_k(S_\tau)$ the set of k-dimensional faces of simplicial complex Δ. Assume that $f_k(\Delta) = |\Delta_k|$, $f(\Delta) = (f_0(\Delta), \ldots, f_d(\Delta))$ and $f(\lambda, \Delta) = \sum_{k=0}^{d} f_k(\Delta)\lambda^k$.

Represent the polynomial $f(\lambda, \Delta)$ as $f(\lambda, \Delta) = \sum_{k \in \mathbf{Z}_+} \gamma_k(\Delta)\lambda^k(1+\lambda)^{d-k}$. The integer sequence $\gamma = (\gamma_0, \gamma_1, \ldots)$ is called (d, n)-*realized* if $\gamma_k = \gamma_k(\Delta)$ for $k = 0, \ldots, d$ and $\gamma_k = 0$ for $k > d$.

For any natural numbers a and i there exists a unique *binomial i-expansion of a*, $a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$, where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. The number $a^{<i>} = \binom{1+a_i}{1+i} + \cdots + \binom{1+a_j}{1+j}$ is called *the i-th pseudopower of a*.

Theorem 1.

1. *If there exists k, such that $\gamma_{k+1} > \gamma_k^{<k>}$ then γ is not realized for any d.*

2. *If* $\gamma_{k+1} \leq \gamma_k^{<k>}$ *for* $k = 1, \ldots, d-1$ (*Macaulay property*) *then* γ *is* $(2d)$*-realized.*

The following theorem allows us to find minimal d, such that the sequence γ is d-realized.

Theorem 2. *The integer sequence* $\gamma = (\gamma_0, \gamma_1, \ldots)$ *is* d*-realized if and only if the following conditions hold:*

1. $\gamma_0 = 1$, $\gamma_i \geq 0$ *for* $i = 1, \ldots, d$ *and* $\gamma_k = 0$ *for integers* $k \geq d$,
2. $\gamma_i \geq \gamma_{d-i} \leq \gamma_{d-i-1}$ *for* $i = 1, \ldots, \lfloor \frac{d}{2} \rfloor$,
3. $\gamma_{i+1} - \gamma_{j-i} \leq (\gamma_i - \gamma_{j+1-i})^{<i>}$ *for* $j = d, \ldots, 2d$ *and* $i = 1, \ldots, \lfloor \frac{j}{2} \rfloor$.

This criterion was reported to here last year [2].

To solve the same problem concerning triangulation of the cone, given by nonhomogeneous system we can find in linear space $\mathbf{Q}^{d+1}$ a matrix B_1, analogous to the matrix B, replace d by $d+1$, and use theorems 1 and 2. This allows us to give a much simpler proof of the criterion, because the induction on d can be applyed.

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Polyhedral complementarity approach to parametric generalized linear exchange model

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The participants' set of the generalized linear exchange model include alongside consumers also firms supplying the market with the additional volumes of commodities. Each firm is characterized by some total financial liability of the production which is to be kept providing a minimal discontent with the production plan. The consumers have some initial money stock and initial stock of products.The purchase and sale of products for all participants (consumers and firms) are carried out by common prices. Choosing a purchase vector of the products, the consumer tends to maximize his linear utility function. The concept of an equilibrium for this model is introduced by analogy with the linear exchange model.

In the parametric version of the model we study the equilibrium change when the consumers money stock is a linear function of one parameter. An finite algorithm based on polyhedral complementarity approach is proposed.

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The relaxation and facets of the graph approximation problem polytope

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In this paper we consider the following problem. Let $K_n = (V, E)$ is a complete graph on n vertices without loops and multiple edges, G - subgraph of K_n. Through $\mu(V)$ denote the family of all subgraphs of K_n, which connected components are cliques. The graph approximation problem is to find a graph $M^* \in \mu(V)$ that minimizes the functional

$$\rho(G, M) = |EG \cup EM| - |EG \cap EM|.$$

This problem is NP-hard in the general case [1]. For this problem, some known polynomially solvable cases [2], constructed estimates of the objective function [3], developed approximate algorithms [4].

We consider the polyhedral structure of the graph approximation problem. Let R^E is space associated with the set E. The vector incidence of subgraph $H \subseteq K_n$ is a vector $x^H \in R^E$ with coordinates $x_e^H = 1$ if $e \in EH$ and $x_e^H = 0$ if $e \notin EH$. Correspondingly, the polytope of this problem is a set

$$P = conv\{x^M \in R^E | M \in \mu(V)\}.$$

Theorem 1. *Polytope P is the convex hull of integer solutions of the system*

$$\begin{gathered} x_{uv} + x_{u\omega} - x_{v\omega} \leq 1 \\ x_{uv} - x_{u\omega} + x_{v\omega} \leq 1 \\ -x_{uv} + x_{u\omega} + x_{v\omega} \leq 1 \\ x_{uv} \geq 0, \end{gathered} \tag{1}$$

where $u, v, \omega \in V$ are possible triples of distinct vertices.

In these terms, the objective function $\rho(G, M)$ for a given G will have the form

$$f(x) = |EG| + \sum_{e \in E\bar{G}} x_e - \sum_{e \in EG} x_e.$$

When solving combinatorial problems occupy a special place the facet inequalities. The support inequality to the polytope called facet inequality, if it generates the maximum dimension on its own face of polytope. Application

the facet inequalities as cutting planes produces an impressive record for large-scale problems. This is due to two main factors. First, each facet inequality present in any linear description of the polytope, and secondly, they are the most "powerful" cutting planes in the geometric sense.

Theorem 2. *Each constraint of the system (1) defines a facet of the polytope P.*

Theorem 3. *Let $u \in V$ and $W \subseteq V \setminus \{u\}$. The linear inequality*

$$\sum_{v \in W} x_{uv} - \sum_{v,w \in W | v \neq w} x_{vw} \leq 1$$

defines a facet of the polytope P.

In addition to these classes of inequalities we have obtained a wider class.

Theorem 4. *Let $U \subset V$ and $W \subseteq V \setminus U$. The linear inequality*

$$\sum_{u \in V} \sum_{v \in W} x_{uv} - \sum_{v,w \in W | v \neq w} x_{vw} \leq \lfloor \frac{(2|U|+1)^2}{8} \rfloor$$

is a support inequality to the polytope P.

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Contradictory convex programs: the residual optimal correction method

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Consider the convex programming (CP) problem

$$\min\{f_0(x) \mid x \in X\}, \tag{1}$$

where $X = \{x \mid f(x) \leq 0\}$, $f(x) = [f_1(x), \ldots, f_m(x)]$, $f_i(x)$ are convex functions defined on $\mathbb{R}^n$ $(i = 0, 1, \ldots, m)$. The CP problems with contradictory constraints $(X = \varnothing)$ represent [1] an important class of improper CP problems (ICPP).

The improper problems often appear because of the errors in the initial data, that is connected with the stability of the optimal elements. Similar problems are an object for the theory of ill-posed optimization problems. This fact makes it possible to consider the standard regularization methods of the ill-posed models for the analysis of improper problems. In this report the possibility of the use of the residual method [2] for the approximation of the ICCP is investigated.

In the residual method the following similar problem is considered along with (1):

$$\min\{\|x\|^2 \mid x \in X \cap M_\delta\}, \tag{2}$$

where $M_\delta = \{x \mid f_0(x) \leq \delta\}$, $\delta \in \mathbb{R}^1$, $\delta \geq f^*$, f^* is the optimal value of problem (1). The problem (2) has a unique optimal point x^*_δ, and it is easy to verify, that sequence x^*_δ converge to the normal solution of the problem (1) as $\delta \to f^*$.

If the set X is empty, then the problem (2) is also the ICCP. In this case we register the restrictions of the problem (2) by a penalty function. There are considered the method of the quadratic penalty function and the exact penalty function method. Then instead (2) there are investigated the problem

$$\min_x\{F_\delta(x, r) = \|x\|^2 + \rho\|f^+(x)\|^2 + \rho_0(f_0(x) - \delta)^{+^2}\}, \tag{3}$$

and the problem

$$\min_x\Big\{\Phi_\delta(x, r) = \|x\|^2 + \rho\sum_{i=1}^{m} f_i^+(x) + \rho(f_0(x) - \delta)^+\Big\}, \tag{4}$$

respectively, where $r = [\rho, \rho_0] \in \mathbb{R}^2$, $r > 0$, $\delta \in \mathbb{R}^1$. The both problems (3), (4) have a unique solution for any r, δ and also for $X = \varnothing$. This reason admits to use the functions $F_\delta(x, r)$ and $\Phi_\delta(x, r)$ for the analysis of the ICCP.

In the report the estimates characterizing the convergence of the solution (3) and (4) to appropriate approximate solutions of the ICCP are given. Besides, we discuss the features of the practical realization of the methods based on the functions from (3) and (4). Two techniques of parameter $\delta = \delta_k$ selection for the convergence of residual method are proposed.

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Circumscribed 2d-ellipsoid and Shor's r-algorithm

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In this report, we discuss the subject of "construction of an algorithm, which would be practically no less efficient than r-algorithm and would be as substantiated as ellipsoid method", initiated by N.Z. Shor in 1982. Ellipsoid method uses space dilation operator $R_\alpha(\xi) = I + (\alpha - 1)\xi\xi^T$ to transform 1d-ellipsoid into a new ball in one space dilation. This 1d-ellipsoid is a minimal volume ellipsoid, containing half-ball in n-dimensional Euclidean space. 1d-ellipsoid does not explain dilation of space, which is used in the Shor's r-algorithms. In fact, it can be done using 2d-ellipsoid [1], whose projection onto the plane is given in Fig. 1.

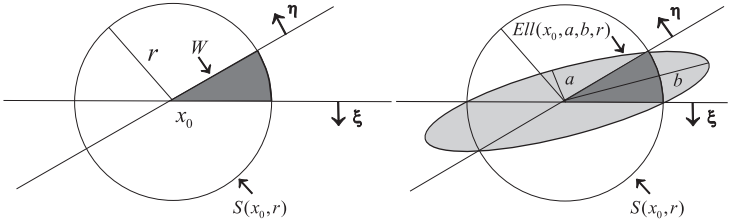

Fig. 1. Projections of the set W and 2d-ellipsoid $Ell(x_0, a, b, r)$

2d-ellipsoid is the minimum volume ellipsoid $Ell(x_0, a, b, r)$ centered at x_0, which contains a convex set $W \in E^n$, resulting from intersection of n-dimensional ball $S(x_0, r) = \{x\colon \|x - x_0\| \le r\}$ with two half-spaces $P(x_0, \xi)$=$\{x\colon (x - x_0, \xi) \le 0\}$ and $P(x_0, \eta)$=$\{x\colon (x - x_0, \eta) \le 0\}$, subject to $-1 < (\xi, \eta) < 0$, $\|\xi\|$=1, $\|\eta\|$=1. 2d-ellipsoid has the following parameters: the length of the semi-axis in the direction $(\xi - \eta)$ is equal to $a = r\sqrt{1 + (\xi, \eta)} < r$; the length of the semi-axis in the direction $(\xi + \eta)$ is equal to $a = r\sqrt{1 - (\xi, \eta)} < r$; semi-axes length in the other $(n - 2)$ directions which are orthogonal to ξ and η, is equal to r. The ratio of 2d-ellipsoid volume to ball volume is equal to $q = (a/r) * (b/r) = \sqrt{1 - (\xi, \eta)^2} < 1$. It decreases when the angle between vectors ξ and η approaches 180 degrees.

Theorem 1. *Let B_k be $n\times n$-matrix, such that $\|B_k^{-1}(x_k-x^*)\|\le r$; g_1 and g_2 are n-dimensional vectors, subject to $(x_k-x^*,g_1)\ge 0$ and $(x_k-x^*,g_2)\ge 0$. If the condition $-\|B_k^T g_1\|\|B_k^T g_2\| < (B_k^T g_1, B_k^T g_2) < 0$ is satisfied and matrix B_{k+1} is recalculated according to the rule*

$$B_{k+1} = B_k R_{\beta_1}\left(\frac{\xi-\eta}{\|\xi-\eta\|}\right) R_{\beta_2}\left(\frac{\xi+\eta}{\|\xi+\eta\|}\right), \quad \xi = \frac{B_k^T g_1}{\|B_k^T g_1\|}, \ \eta = \frac{B_k^T g_2}{\|B_k^T g_2\|},$$

where $\beta_1=\sqrt{1+(\xi,\eta)}$ $\beta_2=\sqrt{1-(\xi,\eta)}$, then matrix B_{k+1} has the following properties: (i) $\|B_{k+1}^{-1}(x_k-x^*)\| \le r$; (ii) $\det(B_{k+1}) = \det B_k\sqrt{1-(\xi,\eta)^2}$; (iii) $(B_{k+1}^T g_1, B_{k+1}^T g_2) = 0$.

Theorem 1 has the following interpretation: Property (i) assures that the point x^* is located within the next 2d-ellipsoid, and property (ii) means that the volume of the ellipsoid decreases in comparison with that of previous ellipsoid. Property (iii) assures use of anti-zigzag method like one used in r-algorithms. This means that subgradients with obtuse angle in current variable space become orthogonal in transformed space, that allows to improve level surfaces of ill-conditioned functions. Coefficients of space dilation in direction of difference of normalized subgradients and in direction of sum of normalized subgradients are defined by angle between subgradients. The obtuser the angle, the larger space dilation coefficient in the direction of difference of two normalized subgradients.

2d-ellipsoid can be used for construction of accelerated ellipsoid methods for a broad class of problems: convex programming problems, finding saddle points of concave-convex functions, special cases of variational inequalties, linear and non-linear complimentarity problems. For these problems, one can guarantee the rate of convergence close to one of r-algorithms. This is confirmed by subgradient methods with space transformation for finding the minimum point of a convex function with a priori knowledge of function value at the minimum point [2]. They proved to be effective while working with ill-conditioned functions.

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New Approach for Solving Nonconvex Optimal Control Problems

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As known, the most of real-life optimization problems turn out to be both nonconvex and dynamic, what provides a huge of difficulties and singularities in studying and moreover in a numeric search for a global solution to the problems.

We consider optimal control problems with quadratic functionals defined by matrices which are indefinite, otherwise the problems with integral and terminal (d.c.) functionals, representable as a difference of two convex functionals (Bolza problems). For this class of optimal control problems we propose, first, special local search methods, consisting in consecutive solving the linearized (w.r.t. the basic nonconvexities at a current iteration) problems, and a study of its convergence [6].

Second, for the optimal control problems under scrutiny Global Optimality Conditions, from which, in particular, the Pontryagin maximum principle follows [4, 5, 10] have been developed. The constructive (algorithmic) property of new optimality conditions is also demonstrated, and examples of nonconvex optimal control problems have been solved by means of global optimality conditions. In these examples we performed an improvement of a feasible control satisfying Pontryagin principle with a corresponding improvement of the cost functional. Finally, employing Pontryagin principle and new Global Optimality Conditions we give a demonstration of construction of a optimal control method and provide for new result on its convergence [7, 8, 9].

Besides a family of Local Search methods special for each kind of nonconvexity was proposed and substantiated, and after that incorporated into Global Search Procedures [3].

Further the number of special nonconvex optimal control test problems has been generated by the procedure the idea of which belong to L.N. Vicente and P.H. Calamai [1, 2].

On this large field of benchmarks problems some of that are of rather high dimension (20 in state and 20 in control) it was conducted a large number of computational experiments which witnessed on the attractive abilities and the promising effectiveness of the developed approach.

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Management and decision making in hierarchical communication networks with discrete flows

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Conceptual approach to creation of the automated information and analytical system of support of decision-making (AIAS SDM) for management of processes the processing and distribution of small discrete of flows (LCL cargo flows, messages, a small lot cargos) in of hierarchical multicommodity networks is considered [1]. The work is performed within the framework of basic research undertaken by the Institute of telecommunications and global information space of NAS of Ukraine on creating toolkit and an information platform (a Portal) for automation of decision-making processes in multicommodity communication networks. The aim is to increase the efficiency of functioning of multicommodity communication networks on the basis of the proposed methodology of mathematical modelling of processes distribution of discrete flows and automation of information and analytical support of decision making in management of flows.

In the report discusses theoretical and methodological bases of construction of distributed hierarchical multicommodity communication networks with the small discrete of flows, including: conceptual basis; principles, concepts and criteria of processing and distribution of flows; methodology of synthesis and the analysis of multilevel structure of a network [2]. For control a nonlinear and nonstationary process to processing and distribution of flows in a network is proposed a methodology of construction a multilevel system AIAS SDM, which functions in real time and opens a new avenues for creation of intellectual methods and information technology in management of complex processes and objects. The methodology is based on an integrated of decision of problems of current planning, operative management and development. Discusses a mathematical models and a software package for the solution of group of the task of current planning. Proves their use as a methodological of basis - nucleus for the decision of problems of operative management by processes of processing and distribution of flows and also for the decision of problems of perspective development of a network. The interrelation of problems of operative management and perspective development with other problems of AIAS SDM is shown. Are given general methodology and techniques, allowing to make a choice in case of occurrence of non-staff of situations in a network and to accept the decision of a optimal managing in conditions of risk of investments in development of a network. Is marked, that the important feature of the offered methodology is

her universality. She allows to carry out mathematical modeling and optimization of functioning also traditional logistic industrial and transport - warehouse systems, which include: a nodes of suppliers of raw material, manufacture of the goods, warehouses and end users.

Implementation of the received results in the activities of state and commercial transport companies will enable get informed decisions important for transport systems tasks: calculate key performance and economic performance of the transport network; improve the functioning of the network in real time by optimizing of schemes the processing and transportation of cargo; operative manage of the processes of processing and transportation of cargos in cases of extraordinary events at the expense of their timely detecting and redirecting flows; to plan a phased development of network, taking into account the commercial risk of investing in the conditions of instability of economy and devaluation of resources; reduce the cost of network design; model various options of a transport network in an interactive mode of optimization and to choose the best option from family of the received results taking into account the chosen function of the purpose and the accepted restrictions.

In the conclusion the basic results are formulated and the directions of further researches are defined.

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Optimal transmission capacity for two-node market

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We examine a two-node competitive market. The goal is to maximize the total welfare and to determine the optimal transmission capacity. Consumers at the node $i = 1, 2$ are described by a non-increasing demand function $d_i(p)$, $d_i(p) \to 0$ as $p \to \infty$. There is a final set of producers A_i at the local market i. Each producer a is characterized by convex cost function $E^a(q)$. Loss coefficient k determines the share of the good, lost under transmission; C is the be transmition capacity.

Strategy of producer a is a non-decreasing supply function $r^a(p)$ that determines the output volume depending on the price p. First, the network administrator computes clearing prices $\overline{p_i}$ for isolated markets. These prices are determined by the equations $\sum_{a\in A_i} r^a(p_i) = d_i p_i, i = 1, 2$. If $1 - k \leqslant \overline{p_2}/\overline{p_1} \leqslant (1 - k)^{-1}$, there is no transmission from one market to the other. Otherwise let $\overline{p_2}/\overline{p_1} \leq (1 - k)^{-1}$. Then the network administrator determines the volume of good v that will be transmitted from the market 1 to market 2. Under a given flow, v the nodal prices $p_1(v)$ and $p_2(v)$ meet the equations $\sum_{a\in A_1} r^a(p_1) = d_1 p_1 + v, \sum_{a\in A_2} r^a(p_2) = d_2 p_2 - (1 - k)v$.

There are two possible outcomes. The first one corresponds to the conditions: $(1-k)^{-1}p_1(v) = p_2$ and $v \leqslant C$. Otherwise $v = C$ and $(1-k)^{-1}p_1(v) < p_2$.

Under assumption of perfect competition, the optimal strategy of each producer is determined by Walrasian supply function $s^a(p) = \arg\max_{q^a}(q^a p - E^a(q^a))$

Let $\widetilde{p_i}(Q), i = 1, 2$ denote the prices corresponding to Walrasian supply functions. Prices $\widetilde{p_i}(0)$ meet the equations $s_i(\widetilde{p_i}) = d_i(\widetilde{p_i}), i = 1, 2$. We assume that $(1 - k)\widetilde{p_2}(0) > \widetilde{p_1}(0)$. If there is a flow from the first market to the second market, the prices satisfy the following conditions

$$s_1(\widetilde{p_1}) = d_1(\widetilde{p_1}) + v, s_2(\widetilde{p_2}) = d_2(\widetilde{p_2}) - (1 - k)v \tag{1}$$

$v \leq C, \widetilde{p_1} = (1 - k)\widetilde{p_2}$ or $v = C, \widetilde{p_1} < (1 - k)\widetilde{p_2}$

Let functions ${\widetilde{p_1}}^0(v)$ and ${\widetilde{p_2}}^0(v)$ be determined by (1). If $v = C$, then $\widetilde{p_i}(C) = {\widetilde{p_i}}^0(C)$.

Theorem 1. *There exists a value of the transmission capacity $\widehat{C}$ for which ${\widetilde{p_1}}^0(\widehat{C}) = (1 - k){\widetilde{p_2}}^0(\widehat{C})$. If $C \geq \widehat{C}$, in equilibrium $v = \widehat{C} < C, \widetilde{p_i}(C) = {\widetilde{p_i}}^0(\widehat{C}), i = 1, 2$. Otherwise, $v = C, \widetilde{p_i}(C) = {\widetilde{p_i}}^0(C), i = 1, 2$ $\widetilde{p_1}(C) < (1 - k)\widetilde{p_2}(C)$*

Without taking into account the construction costs, the welfare $N(C)$ includes the benefit of the network system $T(C)$, consumer surplus $S_i(C)$ and producers profit$P_i(C), i = 1, 2$. The benefit of the network system is determined by $T(C) = \widetilde{p_2}(C)(1-k)C - \widetilde{p_1}(C)C$, if $C < \widehat{C}$. Otherwise it equals zero. Producers at market i get the total profit equal to

$$P_i(C) = \sum_{a \in A_i} ((\widetilde{p_i}(C)s^a(\widetilde{p_i}(C)) - E^a(s^a(\widetilde{p_i}(C)))))$$

.

Consumer surplus at market i is given by $S_i(C) = \int_{\widetilde{p_i}(C)}^{\infty} d_1(p)dp$. Let $B(C)$ denote the costs of the transmission line construction: $B(C) = 0$, if $C = 0$; $B(C) = b_f + b_v(C)$, if $C > 0$, where $b_v(C)$ is a convex and increasing function. The total welfare is $W(C) = N(C) - B(C)$.

Theorem 2. *Function $N(C)$ is concave and increases in C, if $C \leq \widehat{C}$. In addition, $N^{'}(C) = (1-k)\widetilde{p_2}(C) - \widetilde{p_1}(C)$.*

Theorem 3. *The optimal transmission capacity $C^* = 0$, if $(1-k)\widetilde{p_2}(0) - \widetilde{p_1}(0) \leq b_v^{'}(0)$. Otherwise a local maximum C^{*L} is determined by $(1-k)\widetilde{p_2}(C^{*L}) - \widetilde{p_1}(C^{*L}) = b_v^{'}(C^{*L})$. If $W(C^{*L}) > W(0)$ then $C^* = C^{*L}$. Otherwise, $C^* = 0$.*

Cross-boarder capital allocation in Russia

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Most macroeconomic computable general equilibrium models (CGE) describe cross-boarder capital allocation as functions of interest rates, exchange rates, national growth rates and national accounts. A CGE model of an open economy should contain Balance of payments module to represent current account flows (exports and imports of goods and services, balance of salary, dividends, transfers, etc.) and capital account flows (increments and decrements of assets and liabilities). Elaborate study of Russian Balance of payments revealed that common approaches are inapplicable to Russian data. This research presents another way to model Balance of payments based on unique properties of Russian data.

Three aggregated groups of assets ($Lf(t)$ – loans of Russian banks to foreigners, $DIp(t)$ – direct investments from Russia and DIo(t) – other investments from Russia) and four aggregated groups of liabilities ($Df(t)$ – loans for Russian banks from abroad, $Dff(t)$ – loans for Russian enterprises from abroad, $INVp(t)$ – direct investments from abroad, $INVo(t)$ – other investments from abroad) are contemplated. The data reveals a few steady numerical facts:

$DIp(t) = INVp(t)$

$Lf(t) + DIp(t) = Dff(t)$, and after 2008 – $Lf(t) + DIp(t) > Dff(t)$

$Lf(t) + DIp(t) = INVp(t) + INVo(t)$, and after 2008 – $Lf(t) + DIp(t) > INVp(t) + INVo(t)$

$DIo(t) = Dff(t) + INVo(t)$, and after 2008 – $DIo(t) > Dff(t) + INVo(t)$

$Lf(t) + DIp(t)\ Dff(t) + INVo(t)$, and after 2008 – $Lf(t) + DIp(t) > Dff(t) + INVo(t)$

$Lf(t) + DIo(t) = Df(t) + Dff(t)$, and after 2008 – $Lf(t) + DIo(t) > Df(t) + Dff(t)$

$Lf(t) + DIo(t) = Df(t) + INVp(t)$, and after 2008 – $Lf(t) + DIo(t) > Df(t) + INVp(t)$

$DIp(t) + DIo(t) = Dff(t) + INVp(t)$, and after 2008 – $DIp(t) + DIo(t) > Dff(t) + INVp(t)$

$DIo < Df(t) + INVp(t)$, and after 2008 – $DIo(t) = Df(t) + INVp(t)$

$DIp(t) + DIo(t) = Df(t) + Dff(t) + INVo(t)$, and after 2008 – $DIp(t) + DIo(t) > Df(t) + Dff(t) + INVo(t)$

$DIp(t) + DIo(t) = Dff(t) + INVp(t) + INVo(t)$, and after 2008 – $DIp(t) + DIo(t) > Dff(t) + INVp(t) + INVo(t)$

These facts allow constructing the model based on the assumption that a Russian macroeconomic agent Owner allocates capital abroad and reallocates it backward subject to the mentioned constraints. The utility of the agent is a monotonous function of interest payments on foreign loans held by Russian enterprises rDff(t)

$$\int_{t0}^{T} \left(\frac{rDff\,(t)}{rDff0} \right)^{1-\eta} \mathrm{e}^{-\delta\,(t-t0)} dt$$

The data also reveals the following crucial fact

$$DIp(t) + DIo(t) = (DIp(t-1) + DIo(t-1))\,(1 + r_e) + rDff(t)$$

One can choose any number of facts to develop a model. All combinations sum up to over 2000. A special reduction procedure excludes over 1000 systems that provide no solution. Some of the rest are developed via ECOMOD library in MAPLE to achieve a model that reproduces $Lf(t)$, $DIp(t)$, $DIo(t)$, $Df(t)$, $Dff(t)$, $INVp(t)$ from 2004 till 2013.

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About minimax Weber problem in the plane with forbidden gaps

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The problem is to locate n new facilities $X_1, ..., X_n$ in the plane when there are m facilities $P_1, ..., P_m$ already located. There are the forbidden rectangular gaps F_k, which sides are parallel to axes of coordinates, $F = \bigcup F_k$, $k = 1, ..., z$; $w_{ij} \geq 0$ and $v_{jk} \geq 0$ – costs of communications between P_i and X_j, X_j and X_k respectively. We need to locate the facilities $X_1, ..., X_n$ out of the forbidden gaps so as to minimize the most weighed distance between all facilities. The mathematical model is [1]:

$$\max\{\max_{1\leq i\leq m, 1\leq j\leq n} w_{ij}d(P_i, X_j), \max_{1\leq j<k\leq n} v_{jk}d(X_j, X_k)\} \to \min, \quad (1)$$

$$X_j \notin Int\ F,\ j = 1, ..., n, \quad (2)$$

where $d(\cdot, \cdot)$ – some metric, $Int\ F$ – interior of F.

In the report problem (1)–(2) with rectangular metric is considered. It is proved that for search of the optimum sufficient to consider a subset of admissible solutions. The variant of algorithm branch and bounds is developed. Computing experiment in comparison of efficiency of the algorithm and the solving of the problem by means of integer linear programming model and package IBM ILOG CPLEX is represented.

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Technique of computational investigating of nonconvex optimal control problems in systems not solvable with respect to the derivative

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We propose a set of reductions, allowing to reduce the initial extreme models in the form of ordinary differential equations not solvable with respect to the derivative [1–6] to optimal control problems in the system of differential equations in the Cauchy form. It is carried out the numerical experiments to study the properties of proposed reductions in the problem of stabilization of the pendulum in the linear and nonlinear formulation and others. We have used software OPTCON-A for solving considered optimization problems [7]. Implemented computational experiments have demonstrated the effectiveness of the proposed approach.

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Dual two-stage affine scaling method for linear semidefinite programming problem

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In this paper we consider the variant of dual affine scaling method for solving the linear semidefinite programming problem

$$\min\ C \bullet X, \qquad A_i \bullet X = b^i, \quad 1 \le i \le m, \quad X \succeq 0, \tag{1}$$

where C, X and A_i, $1 \le i \le m$, are symmetric matrices of order n, the inequality $X \succeq 0$ indicates that X must be a semidefinite matrix. The operator $\bullet$ denotes the standard inner product between two matrices. The dual problem to (1) has the form

$$\max\ b^T u, \quad \Sigma_{i=1}^m u^i A_i + V = C, \quad V \succeq 0, \tag{2}$$

with $b = [b^1, \ldots, b^m]$ and $u = [u^1, \ldots, u^m]$.

In [1] the dual affine scaling method is proposed. This method converges to solution of (2) with the linear rate. But the convergence is only local and rather slow since the step size applying in the method is taken constant and sufficiently small. In [2] the modification of the method based on the special relaxation of problem (2) is worked out. In this variant of the method the step size is chosen on the base of steepest descent. Thus, the method converges globally at the feasible set. In some sense the behavior of the method is similar to dual simplex method.

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Application of the FAD-methodology to the solution of the inverse problem

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The classical heat equation is used for the description and mathematical modeling of a wide variety of thermal processes. In this equation the density of the substance, its specific heat capacity and thermal conductivity are considered known functions of the coordinates and temperature. Specifying of additional boundary conditions allows to determine the dynamics of the temperature field in the studied substance.

However, not all the properties of the substance are known. Quite often a situation is encountered when the thermal conductivity of the substance depends only on the temperature and this dependence is unknown. In this case arises the problem of determining the dependence of the thermal conductivity of the material on the temperature using the results of experimental observation of the dynamics of the temperature field. The same problem occurs in the case when a complex thermal process is described using a simplified mathematical model. For example, when describing the process of heat diffusion in composite materials, where an important role plays the heat transfer due to radiation, a simplified model is used, which assumes the absence of radiative heat transfer, but its effect is modeled by an effective coefficient of thermal conductivity, which is to be determined.

In this work we consider the problem of determination of a temperature-dependent thermal conductivity coefficient of the material in a one-dimensional problem. Dynamics of the temperature field is determined by solving the following initial-boundary value problem

$$\rho C\frac{\partial T(x,t)}{\partial t}=\frac{\partial}{\partial x}\left(K_{eff}(T)\frac{\partial T(x,t)}{\partial x}\right)+f(x,t),\quad (0<x<L,\ 0<t<\Theta),$$

$$T(x,0)=T_0(x),\qquad (0\le x\le L),$$

$$T(0,t)=T_1(t),\qquad T(L,t)=T_2(t),\qquad (0\le t\le\Theta),$$

with a specially chosen convective coefficient of thermal conductivity K_{eff}. This coefficient should be chosen so that the temperature field $T(x,t)$, obtained by solving the initial-boundary value problem, was very close to the field $\widehat{T}(x,t)$, which is obtained experimentally. As a measure of deviation of the solutions

the following value can be used

$$\Phi(K_{eff}(T)) = \alpha \cdot \int_0^{\Theta} \int_0^{L} \left[T(x,t) - \widehat{T}(x,t)\right]^2 \cdot \mu(x,t)dxdt +$$

$$+\beta \cdot \int_0^{\Theta} \left[K_{eff}(T(0,t)) \cdot \frac{\partial T}{\partial x}(0,t) - P(t)\right]^2 dt,$$

where $\mu(x,t) \geq 0$ is a given weight function and $P(t)$ is the known heat flux on the left boundary of the domain.

The problem considered in this paper is to determine the unknown function $K_{eff}(T)$.

The formulated optimal control problem was solved numerically. The unknown function $K_{eff}(T)$ was approximated by a continuous piecewise linear function. The cost functional was minimized using the gradient method. To calculate the gradient of the functional the effective Fast Automatic Differentiation technique was used.

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Investigation of the Optimal Control Problem for Metal Solidification in a New Statement for Objects with Complex Geometry

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An optimal control problem of the metal crystallization process is investigated for a new model of the furnace. The solidification of the metal can be described as follows. Molten metal is poured into a mold. The cooling of the mold and the metal inside it occurs due to the interaction of the object with its surroundings. For this purpose is used a special setup consisting of a melting furnace, inside which the object moves, and a cooler which is a large tank filled with liquid aluminum. The mold with molten metal is being immersed into the coolant. Liquid aluminum has a relatively low temperature and thus proceeds the crystallization of metal. On the other hand, the object gains heat from the furnace walls, which prevents the solidification process from proceeding too fast.

The velocity of the mold relative to the furnace is a parameter that has a large effect on the evolution of the solidification front in the metal. We use it as a control function. To find a control function satisfying the imposed technological requirements, we formulate an optimal control problem for the metal solidification process. This problem consists in choosing a regime of metal cooling and solidification in which the solidification front has a preset shape (it is desirable that the front be a plane orthogonal to the vertical axis of the object) and moves at a speed close to the preset one.

The object under study has a complex geometrical shape. Due to the complex geometry of the object, we had to further improve the algorithm developed for computing an object of the simplest shape, namely, a parallelepiped. First, at the interface of different parts of the object, there arose new elementary cells of complex geometry that were not encountered earlier. As a result, we had to modernize the algorithm for computing the heat balance in the system. Second, due to the new geometry of the object, the algorithm for computing the level of liquid aluminum in the cooling bath had to be improved. Finally, for the same reason, the algorithm for computing the heat flux to the object caused by the thermal radiation of the furnace and the surface of the liquid aluminum was modified. This is associated with the fact that some parts of the object surface are now shaded by others.

In the old setup model the furnace was modeled by two vertical parallel walls heated to a prescribed relatively high temperature. In the new setup model, the

sidewalls of the furnace are spanned by a horizontal wall (a roof) heated to a given temperature. One of the mechanisms of heat transfer in the given problem is thermal radiation (thermal energy of the body is lost due to its own radiation and energy is gained due to radiation of the surrounding liquid aluminum, the furnace walls, and the surface of the liquid aluminum). Moreover, we have to take into account the heat exchange between the liquid aluminum and the object due to conduction. In the new statement, the energy gained due to radiation of the upper wall of the furnace (the roof) is additionally taken into account.

We consider three versions of the new model. They differ in the heat transfer laws used on the outer surface of the mold. In the first version all walls of the mold gain heat from the furnace, liquid aluminum, and the roof and simultaneously release heat due to conduction and radiation. In the second version we consider a mold with two heat-insulated sidewalls (on the sides where there are no furnace walls). The total heat flux on these walls is zero. In the third version of the model, these sidewalls are assumed to be made of a heat-insulating material that loses its insulation property in liquid aluminum. This version describes the situation when several molds are located in a line in the furnace. The part of a mold lying outside of the coolant is heat-insulated, while the part located in the coolant interacts with the surroundings according to the same laws as in the old model.

For all three versions of the new model we solved a series of optimal control problems for the metal solidification process. The problems were solved numerically using the gradient methods. To calculate the gradient of the cost function the Fast Automatic Differentiation methodology was used.

Based on the studies performed, we conclude that the roof has a considerable influence on the evolution of the solidification process. The new model of the furnace leads to a much better crystallization process.

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On the number of iterations in extragradient methods

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The extragradient methods [1,2,3] are an effective tools for the variational inequalities solving. They require less conditions than the gradient methods need for convergence. It is known, that the monotone convergence in the norm to one of the solutions of the variational inequalities for the extragradient methods. It is known, that convergence extragradient method with the rate of a geometric progression for variational inequalities with the bilinear functional [1]. A finite number of steps for extragradient method for solving variational inequalities with the bilinear functional under some additional conditions is proved [4].

The article deals with problem of the number of iterations one-step [1,2] and double-step [3,5] extragradient methods for solving variational inequalities for different classes of problems [6,7].

To solve *the variational inequality* it is means, that to find a vector $z^* \in \Omega$ that satisfies the following conditions:

$$\langle H(z^*), z - z^* \rangle \geq 0, \quad \forall\, z \in \Omega, \tag{1}$$

where $H : \mathbb{R}^n \to \mathbb{R}^n$, Ω – a convex closed set $\Omega \subset \mathbb{R}^n$, $z^* \in \Omega^*$ – set of solutions for the variational inequality, $\Omega^* \subset \Omega$.

Let $H(z)$ is monotone operator under the Lipschitz condition with a constant $L > 0$. The equations

$$\tilde{z}^k = P_\Omega(z^k - \alpha H(z^k)), \quad z^{k+1} = P_\Omega(z^k - \alpha H(\tilde{z}^k)), \tag{2}$$

$$\bar{z}^k = P_\Omega(z^k - \alpha H(z^k)),\ \tilde{z}^k = P_\Omega(\bar{z}^k - \alpha H(\bar{z}^k)),\ z^{k+1} = P_\Omega(z^k - \alpha H(\tilde{z}^k)) \tag{3}$$

are the recurrence relationships for the one-step [1] and two-step [3] extragradient method for solving the variational inequalities (1).

The convergence is provided by the value of the step α of the one-step (2) and two-step (3) extragradient method for solving $z^* \in \Omega^*$ variational inequalities (1), where α satisfies the following conditions $0 < \alpha < \frac{1}{\sqrt{2}L}$ for the one-step method [1] and which satisfies the following conditions $0 < \alpha < \frac{1}{\sqrt{3}L}$ for the two-step method [3].

The sharpness condition [7] is condition which provide convergence of a computational scheme (2) and (3) in a finite number of iterations. And it performed for the solutions z^* of the variational inequality with $\gamma > 0$:

$$\langle H(z), z - z^*(z) \rangle \geq \gamma \|z - z^*(z)\|, \quad \forall\, z \in \Omega, \quad z^*(z) = P_{\Omega^*}(z). \tag{4}$$

The question of convergence in a finite number of steps for linear programming problem discussed in [4] in case of one-step extragradient method in a more strict sharpness condition $\langle H(z), z - z^* \rangle \geq \gamma \|z - z^*\|$, which assumes the exist the unique solution $z^* \in \Omega$.

As an example were reviewed problems of linear programming and test problem from [7,8].

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